Tight sets in finite geometry

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Outline

tight sets in generalized quadrangles

Definition (S.E. Payne, 1987)

A point set A of a finite generalized quadrangle is *tight* if on average, each point of A is collinear with the maximum number of points of A

Theorem (S.E. Payne, 1973)

Let A be a tight set of a generalized quadrangle $\mathcal{S}$ Then there exists a number $x > 0$ such that P is collinear with exactly x points of A when $P \notin A$ and P is collinear with exactly $s + x$ points when $P \in A$.

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tight sets in generalized quadrangles

An x-tight set *behaves combinatorially* and the disjoint union of x lines of the generalized quadrangles

Finite classical polar spaces

- $V(d+1, q)$: $d+1$-dimensional vector space over the finite field $\mathrm{GF}(q)$.
- f: a non-degenerate sesquilinear or non-singular quadratic form on $V(d+1, q)$.

Definition

A *finite classical polar space* associated with a form f is the geometry consisting of subspaces of $\mathrm{PG}(d, q)$ induced by the totally isotropic sub vector spaces with relation to f.

Finite classical polar spaces

A polar space contains points, lines, planes, etc. of the ambient projective space.

Definition

- The *generators* of a polar space are the subspaces of maximal dimension.
- The *rank* of a polar space is the vector dimension of its generators
- For a point P, the set $P^{\perp}$ of points of $\mathcal{S}$ collinear with P is the intersection of the tangent hyperplane at P with $\mathcal{S}$.

Finite classical polar spaces

flavours: orthogonal polar spaces: quadrics; symplectic polar spaces; hermitian polar spaces.

polar space	rank	form
$Q(2n, q)$	n	$x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n}$
$Q^+(2n+1, q)$	$n+1$	$x_0x_1 + \ldots + x_{2n}x_{2n+2}$
$Q^-(2n+1, q)$	n	$f(x_0, x_1) + x_2x_3 + \ldots + x_{2n}x_{2n+2}$
$W(2n+1, q)$	$n+1$	$x_0y_1 + y_1x_0 + \ldots x_{2n}y_{2n+1} + x_{2n+1}y_{2n}$
$H(2n, q^2)$	n	$x_0^{q+1} + \ldots x_{2n}^{q+1}$
$H(2n+1, q^2)$	$n+1$	$x_0^{q+1} + \ldots x_{2n+1}^{q+1}$

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polar space	rank	form
$\mathrm{Q}(2n, q)$	n	$x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n}$
$\mathrm{Q}^+(2n+1, q)$	$n+1$	$x_0x_1 + \ldots + x_{2n}x_{2n+2}$
$\mathrm{Q}^-(2n+1, q)$	n	$f(x_0, x_1) + x_2x_3 + \ldots + x_{2n}x_{2n+2}$
$\mathrm{W}(2n+1, q)$	$n+1$	$x_0y_1 + y_1x_0 + \ldots x_{2n}y_{2n+1} + x_{2n+1}y_{2n}$
$\mathrm{H}(2n, q^2)$	n	$x_0^{q+1} + \ldots x_{2n}^{q+1}$
$\mathrm{H}(2n+1, q^2)$	$n+1$	$x_0^{q+1} + \ldots x_{2n+1}^{q+1}$

Finite classical polar spaces: some examples

space	rank	# points	# generators
$Q(4,q)$	2	$(q^2+1)(q+1)$	$(q^2+1)(q+1)$
$Q(6,q)$	3	$(q^3+1)(q^2+1)(q+1)$	$(q^3+1)(q^2+1)(q+1)$
$Q^-(5,q)$	2	$(q^3+1)(q+1)$	$(q^3+1)(q^2+1)$
$Q^+(5,q)$	3	$(q^2+1)(q^2+q+1)$	$2(q^2+1)(q+1)$

Strongly regular graphs

Definition

Let $\Gamma = (X, \sim)$ be a graph, it is strongly regular with parameters (n, k, λ, μ) if all of the following holds:

(i) The number of vertices is n.

(ii) Each vertex is adjacent with k vertices.

(iii) Each pair of adjacent vertices is commonly adjacent to λ vertices.

(iv) Each pair of non-adjacent vertices is commonly adjacent to μ vertices.

Adjacency matrix

Let $\Gamma = (X, \sim)$ be a srg(n, k, λ, μ).

Definition

The adjacency matrix of Γ is the matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$

$$a_{ij} = \begin{cases} 1 & i \sim j \\ 0 & i \not\sim j \end{cases}$$

Theorem (proof: e.g. Brouwer, Cohen, Neumaier)

The matrix A satisfies

$$A^2 + (\mu - \lambda)A + (n - k)I = \mu J$$

Eigenvalues and eigenspaces

Corollary

The matrix A has three eigenvalues:

$$k, \tag{1}$$

$$r = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} > 0, \tag{2}$$

$$s = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2} < 0; \tag{3}$$

and furthermore

$$\mathbb{C}^n = \langle j \rangle \perp V_+ \perp V_-.$$

Relations on the parameters

Lemma

$$n\mu = (k-r)(k-s), \tag{4}$$

$$rs = \mu - k, \tag{5}$$

$$k(k-\lambda-1) = (n-k-1)\mu. \tag{6}$$

Finite classical polar spaces and strongly regular graphs

Definition

Let $\mathcal{S}$ be a finite classical polar space. Let V be the set of points. Define the relation $\sim$ on two different points of $\mathcal{S}$ as follows: $P \sim Q$ if and only if P and Q are collinear in $\mathcal{S}$, and $P \not\sim P$. The graph $\Gamma = (V, \sim)$ is called the *point graph* of $\mathcal{S}$.

Lemma

The point graph of a finite classical polar space is a strongly regular graph

An example

Consider $\mathcal{S} = \mathrm{Q}(4, q)$ ($x_0^2 + x_1x_2 + x_3x_4$, rank 2). The parameters of the point graph are:

- $n = (q^2 + 1)(q + 1)$
- $k = q(q + 1)$
- $\lambda = q - 1$
- $\mu = q + 1$

The eigenvalues apart from k are

- $r = q - 1$
- $s = -q - 1$

Geometrical definition

Let $\mathcal{S}$ be a finite classical polar space of rank r over the finite field $\mathrm{GF}(q)$. Denote by $\theta_n(q) := \frac{q^n-1}{q-1}$ the number of points in an $n-1$-dimensional projective space.

Definition

An *m-ovoid* is a set $\mathcal{O}$ of points such that every generator of $\mathcal{S}$ meets $\mathcal{O}$ in exactly m points.

Definition

An *i-tight set* is a set $\mathcal{T}$ of points such that

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i\theta_{r-1}(q) + q^{r-1} & \text{if } P \in \mathcal{T} \\ i\theta_{r-1}(q) & \text{if } P \notin \mathcal{T} \end{cases}$$

Graph theoretical definition

Let Γ be the point graph of a finite classical polar space. Any vector $\chi \in \mathbb{C}^n$ defines a weighted point set of $\mathcal{S}$. Denote the all-one vector by j.

Definition (after Delsarte)

- A vector $\chi \in \langle j \rangle \perp V_-$ a *weighted ovoid.*
- A vector $\chi \in \langle j \rangle \perp V_+$ a *weighted tight set.*

An inequality

Lemma (Delsarte)

Let Γ be an srg(n, l, λ, μ) with eigenvalues r, s different from k. Let $\chi \in \mathbb{C}^n$. Then

$$(j\chi^\top)^2 k + s(n\chi\chi^\top - (j\chi^\top)^2) \leq n\chi A\chi^\top$$
$$\leq (j\chi^\top)^2 k + r(n\chi\chi^\top - (j\chi^\top)^2).$$

Lemma

- *Let χ be a weighted ovoid. Then the first inequality becomes an equality*
- *Let χ be a weighted tight set. Then the second inequality becomes an equality*

The most elementary tight set

Lemma

Let $\mathcal{S}$ be a finite classical polar space. Let χ be the characteristic vector of a generator of $\mathcal{S}$. Then χ is a tight set.

In graph theoretic terms, a generator is a *clique*, i.e. $\chi A \chi^T = x(x-1)$ (where x is the number of vertices in the clique).

Corollary

Let χ be the characteristic vector of a clique and a tight set. Then $j\chi^T = 1 - \frac{k}{s}$.

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Ovoids

Corollary

Let χ *represent a* co-clique, *i.e.* $\chi A \chi^T = 0$. *If* χ *is also an ovoid, then* $j\chi^T = \frac{ns}{s-k}$.

How to define the parameters?

Lemma

- *Let χ be a weighted ovoid. It is called a* weighted m-ovoid *if* $j\chi^T = m\frac{ns}{s-k}$.
- *Let χ be a weighted tight set. It is called a* weighted i-tight set *if* $j\chi^T = i(1-\frac{k}{s})$.

How m-ovoids and i-tight sets meet

Theorem

Let χ be a weighted m-ovoid. Let ψ be a weighted i-tight set. Then

$$\chi\psi^\top = mi.$$

Ovoids

Definition (Tits 1962)

Consider the projective space $\mathrm{PG}(d, K)$, K any field. An ovoid is a set $\mathcal{O}$ of points such that the tangent lines at any point $P \in \mathcal{O}$ is the set of lines through P in a hyperplane of $\mathrm{PG}(d, K)$.

- Ovoids of projective spaces are rare: they only exist in dimensions 2 and 3.
- An ovoid of $\mathrm{PG}(3, q)$, q even yields an ovoid of $\mathrm{W}(3, q)$, q even, and vice versa.
- Ovoids of polar spaces are defined for the first time in 1972 by J.A. Thas in the geometrical way.
- Ovoids of polar spaces are rare: they only occur in *low* rank.

Existence and non-existence of ovoids of polar spaces

- open cases: existence of ovoids of $\mathrm{H}(5, q^2)$, $\mathrm{Q}(6, q)$, $q = p^h$, $3 \neq p$ prime, $h > 1$.
- partially open cases: existence of ovoids of $\mathrm{Q}^+(2n+1, q)$ and $\mathrm{H}(2n+1, q^2)$.

Older results revisited

Theorem (J.A. Thas, 1981)

The polar spaces $\mathrm{Q}^-(5,q)$, $\mathrm{H}(4,q^2)$ *and* $\mathrm{W}(5,q)$ *have no ovoids.*

Proof (for $\mathrm{Q}^-(5,q)$).

- Assume that $\mathcal{O}$ is an ovoid of $\mathrm{Q}^-(5,q)$
- Choose $P, Q \in \mathcal{O}$, $l := \langle P, Q \rangle$
- Count pairs (R,S), $R \in l \setminus \mathcal{O}$, $S \in \mathcal{O} \setminus \{P,Q\}$, $R \in S^\perp$.
- This counting yields $(q-1)(q^2+1) = q^3-1$, a contradiction

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Older results revisited

Lemma

Let $P, Q \in \mathrm{Q}^-(5,q)$ *be two non-collinear points. Then* $q\chi_{\{P,Q\}} + \chi_{\{P,Q\}^\perp}$ *is a weighted* $(q+1)$*-tight set.*

Older results revisited

Theorem

The polar space $\mathrm{Q}^-(5, q)$ *has no ovoids*

Proof.

- Assume that $\mathcal{O}$ is an ovoid of $\mathrm{Q}^-(5, q)$. Choose $P, Q \in \mathcal{O}$.
- Let $\chi_T := q\chi_{\{P,Q\}} + \chi_{\{P,Q\}^\perp}$, then $\chi_T \cdot \chi_\mathcal{O} = q + 1$,
- Observe on the other hand that $\chi_T \cdot \chi_\mathcal{O} = 2q$, a contradiction

□

There is a *similar* proof for the non-existence of ovoids of $\mathrm{H}(4, q^2)$ and $\mathrm{W}(5, q)$.

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Older results revisited

Theorem (S.E. Payne and J.A. Thas, 1984)

The polar space $\mathrm{W}(3, q)$ *has ovoids if and only if q is even.*

Lemma

Let l *be a line of* $\mathrm{PG}(3, q) \setminus \mathrm{W}(3, q)$. *Then* $l \cup l^{\perp}$ *is a 2-tight set.*

Older results revisited

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Let l be a line of $\mathrm{PG}(3, q) \setminus \mathrm{W}(3, q)$. Then $l \cup l^{\perp}$ is a 2-tight set.

Older results revisited

Proof.

- Assume that $\mathcal{O}$ is an ovoid of $\mathrm{W}(3, q)$.
- Consider any line l of $\mathrm{W}(3, q)$, consider $P := l \cap \mathcal{O}$, $R \in l \in \setminus \{P\}$, $S \in l \setminus \{R, P\}$.
- The q lines of $\mathrm{W}(3, q)$ on R different from l lie in a plane π. Their sum is a weighted q-tight set T
- The q lines m_i of π on S different from l are not lines of $\mathrm{W}(3, q)$: we obtain q 2-tight sets.
- Observe that the lines m_i partition the set $T \cap \mathcal{O}$ and that each line contains 0 or 2 points of $\mathcal{O}$.
- This yields $2 \mid q$.

□

Older results revisited

Theorem (A. Blokhuis, G.E. Moorehouse, 1995)

The hyperbolic quadric $\mathrm{Q}^+(2n+1,q)$, $q=p^h$, $n \geq 3$ *has no ovoids if*

$$p^n > \binom{2n+p}{2n+1}^2 - \binom{2n+p-1}{2n+1}^2.$$

Theorem (G.E. Moorehouse, 1996)

The hermitian variety $\mathrm{H}(2n+1,q^2)$, $q=p^h$, $n \geq 2$ *has no ovoids if*

$$p^{2n+1} > \binom{2n+p}{2n+1}^2 - \binom{2n+p-1}{2n+1}^2.$$

Older results revisitedt

Theorem (J. Bamberg, JDB, F. Ihringer)

No ovoids of $\mathrm{Q}^+(9, q)$, *q even, exist.*

Older results revisited

Theorem (JDB, Klaus Metsch)

The hermitian variety $\mathrm{H}(5,4)$ *has no ovoids.*

Theorem (O'Keefe, Thas)

The parabolic quadric $\mathrm{Q}(6,q)$*,* q *prime, has no ovoids*

An improvement to an existing result

Theorem (A. Klein, 2004)

The hermitian variety $\mathrm{H}(2d-1,q^2)$ *has no ovoid if* $d > q^3+1$.

Theorem (J. Bamberg, JDB, F. Ihringer)

The hermitian variety $\mathrm{H}(2d-1,q^2)$ *has no ovoid if* $d > q^3-q^2+2$.

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Theorem (J. Bamberg, JDB, F. Ihringer)

The hyperbolic quadric $\mathrm{Q}^+(2d-1, q^2)$ *has no ovoid if* $d > q^2 - q + 3$.

Open cases

non-existence of ovoids of $\mathrm{H}(5, q^2)$, $\mathrm{Q}(6, q)$, $q = p^h$, $3 \neq p$ prime, $h > 1$, $\mathrm{Q}^+(7, q)$ for certain values of q, $\mathrm{Q}^+(9, q)$.

History of Cameron-Liebler line classes

- 1982: Cameron and Liebler studied irreducible collineation groups of $\mathrm{PG}(d, q)$ having equally many point orbits as line orbits
- Such a group induces a symmetrical tactical decomposition of $\mathrm{PG}(d, q)$.
- They show that such a decomposition induces a decomposition with the same property in any 3-dimensional subspace.
- They call any line class of such a tactical decomposition a "Cameron-Liebler line class"
- A CL line class is characterized as follows: $\mathcal{L}$ is a CL class with parameter x if and only if $|\mathcal{L} \cap \mathcal{S}| = x$ for any spread $\mathcal{S}$.

History of Cameron-Liebler line classes

- trivial examples: $\mathrm{Star}(P)$, $\mathrm{Line}(\pi)$, union and complements

Conjecture

The only Cameron-Liebler line classes are the trivial examples

Theorem (A. Bruen, K. Drudge, 1999)

Let q be odd, there exists a Cameron-Liebler line class with parameter $\frac{q^2+1}{2}$.

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Non-existence results

Apart from many non-existence results for small parameter, the most general and most recent result is the following.

Theorem (A.L. Gavrilyuk, K. Metsch, 2014)

Let $\mathcal{L}$ be a CL line class with parameter x. Let n be the number of lines of $\mathcal{L}$ in a plane. Then

$$\binom{x}{2} + n(n - x) \equiv 0 \pmod{q+1}$$

- Input (Morgan Rodgers, May 2011): there exist Cameron-Liebler line classes with parameter $x = \frac{q^2-1}{2}$ for $q \in \{5, 9, 11, 17, \ldots\}$.
- They all are stabilized by a cyclic group of order $q^2 + q + 1$.
- Question: are these member of an infinite family?
- Through Klein-correspondence: a Cameron-Liebler line class with parameter x is an x-tight set of $Q^+(5, q)$.

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- They all are stabilized by a cyclic group of order $q^2 + q + 1$.
- Question: are these member of an infinite family?
- Through Klein-correspondence: a Cameron-Liebler line class with parameter x is an x-tight set of $\mathrm{Q}^+(5, q)$.

The construction of the infinite family

- We are looking for a vector χ_T such that

$$(\chi_T - \frac{x}{q^2+1}j)A = (q^2-1)(\chi_T - \frac{x}{q^2+1}j)$$

The construction of the infinite family

- Using the cyclic group of order $q^2 + q + 1$:

$$(\chi'_T - \frac{x}{q^2 - 1} j')A' = (q^2 - 1)(\chi'_T - \frac{x}{q^2 - 1} j')$$

The construction of the infinite family

- Using the cyclic group of order q^2+q+1:

$$(\chi'_T - \frac{x}{q^2-1} j')B = (q^2-1)(\chi'_T - \frac{x}{q^2-1} j')$$

- Assume that $q \not\equiv 1 \pmod 3$ then all orbits have length q^2+q+1, this induces a *tactical decomposition* of A'

The construction of the infinite family

Definition

Let $A = (a_{ij})$ be a matrix A partition of the row indices into $\{R_1, \ldots, R_t\}$ and the column indices into $\{C_1, \ldots, C_{t'}\}$ is a *tactical decomposition* of A if the submatrix $(a_{p,l})_{p \in R_i, l \in C_j}$ has constant column sums c_{ij} and row sums r_{ij} for every (i, j).

- the matrix $B = (c_{ij})$.

The construction of the infinite family

Theorem (Higman–Sims, Haemers (1995))

Suppose that A can be partitioned as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}$$

with each A_{ii} square and each A_{ij} having constant column sum c_{ij}. Then any eigenvalue of the matrix $B = (c_{ij})$ is also an eigenvalue of A.

The construction of the infinite family

- We use a description of $\mathrm{Q}^+(5, q)$ in $\mathrm{GF}(q^3) \times \mathrm{GF}(q^3)$.
- Assuming that $q \equiv 1 \pmod 4$, we have control on the entries of the matrix B, and, it turns out that B is a *block circulant* matrix!
- Now we have the eigenvector we are looking for, and also yields the full symmetry group of the tight set.

The infinite family

Theorem (JDB, J. Demeyer, K. Metsch, M. Rodgers)

There exist a CL line class of $\mathrm{PG}(3,q)$, $q \equiv 5, 9 \pmod{12}$ *with a symmetry group of order* $3\frac{q-1}{2}(q^2+q+1)$.

The same infinite family has been found by K. Momihara, T. Feng and Q. Xiang, independently.

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Short and incomplete overview

- CL line classes: most recent non-existence results: L. Beukemann, A. Gavrilyuk, K. Metsch, A.L. Mogilnykh; an infinite family with parameter $\frac{q^2+1}{2}$, $q \leq 5$, different form the Bruen-Drudge example (A. Cossidente and F. Pavese).
- Construction results on tight sets of finite classical polar spaces: A. Cossidente and F. Pavese, e.g. tight sets of $\mathrm{W}(5, q)$
- Results on tight sets of other geometries: J. Bamberg, T. Penttila, S. Kelly, M. Law, J. Schillewaert, A. Devillers (tight sets of (non-classical) GQs)
- ...

Short and incomplete overview

- ...
- Construction of tight sets in other geometries: related to quadrics (B. De Bruyn, I. Cardinali), and partial quadrangles (J. Bamberg, F. De Clerck and N. Durante)
- Characterisation results: assume that T is a non weighted x-tight set of a polar space $\mathcal{S}$. What is the bound n on x such that $x < n$ implies that T is the disjoint union of generators: recent results of K. Metsch.

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