On the Index Coding and Caching Problems

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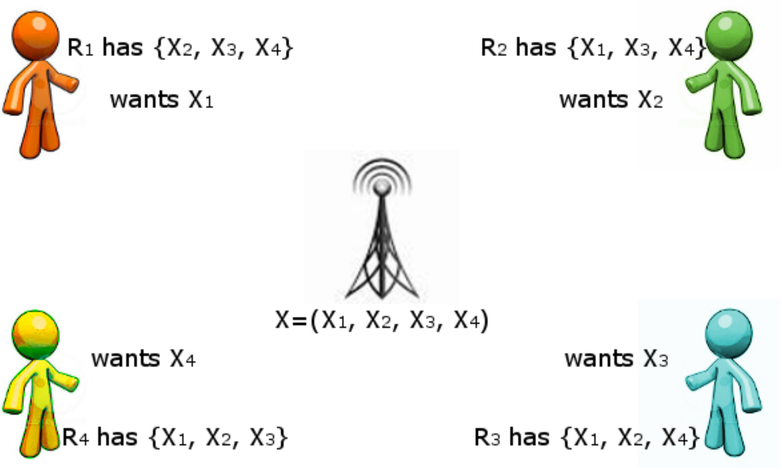
Outline

- The Index Coding with Side Information Problem
- Error Correction in the Index Coding Problem
- Generalizations
  - Coded-Side Information
  - Matrix Channels
- The Coded-Caching Problem
  - Connections to Index Coding
  - Connections to Rank-Distance Codes

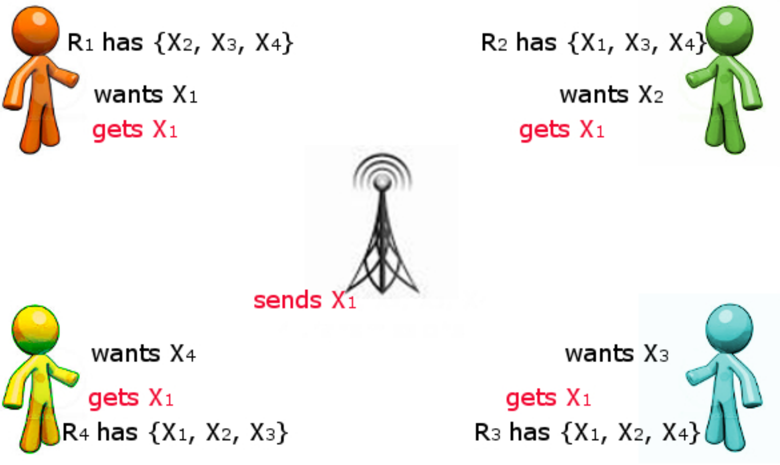
Index Coding with Side Information

- introduced in 2006 by Bar-Yossef, Birk, Jayram & Kol
- has applications to video-on-demand and wireless networks
- equivalent problem to Network Coding
- many approaches to problem from graph theory

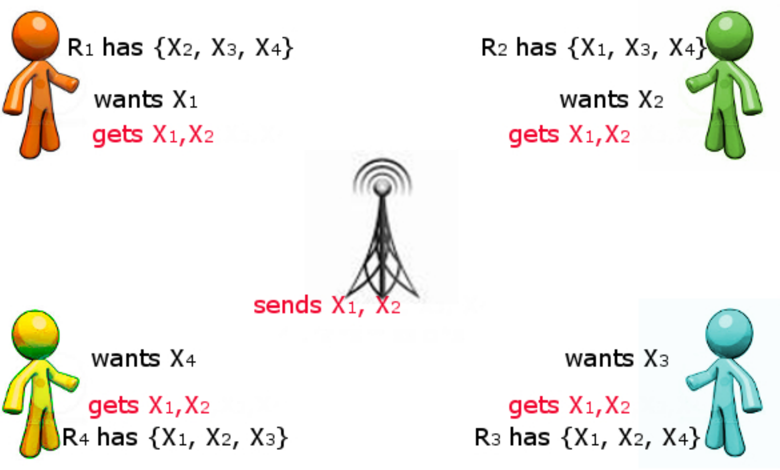
Index Coding with Side Information



Index Coding with Side Information



Index Coding with Side Information



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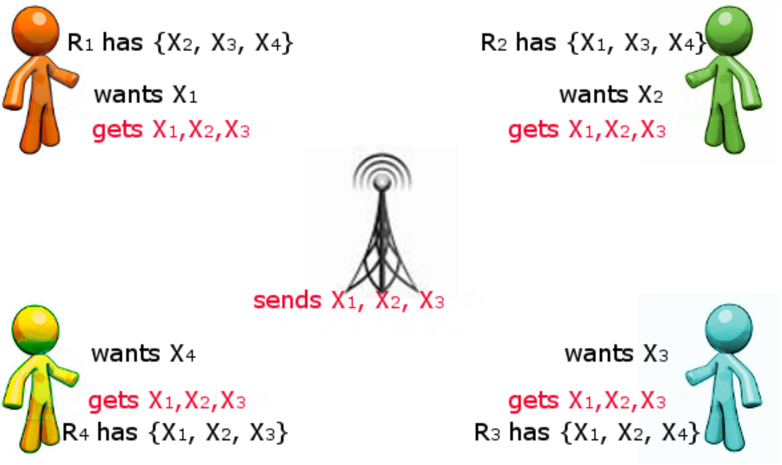
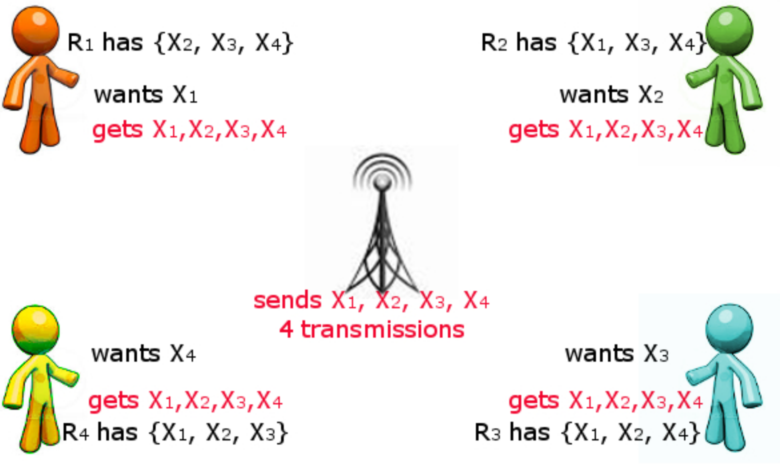
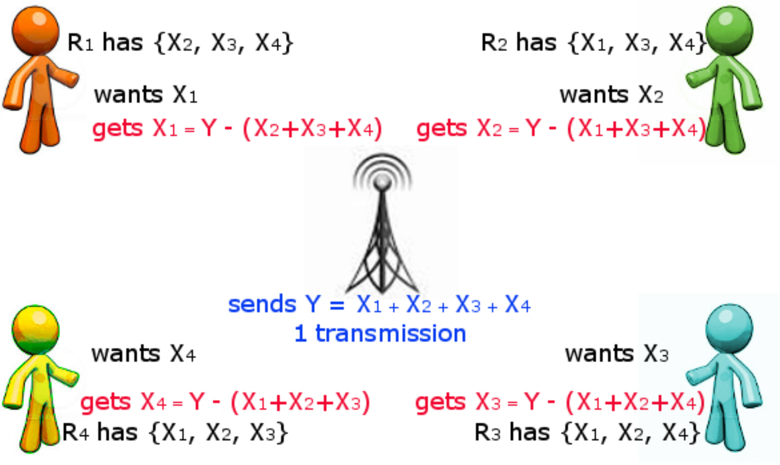


Index Coding with Side Information



Index Coding with Side Information



Index Coding with Side-Information (Original Formula)

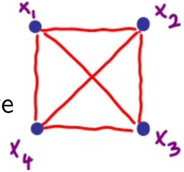
- The sender has a file X split into n packets $X = [X_1, ..., X_n] \in \mathbb{F}_q^n$.
- There are n users $\{1, ..., n\}$.
- User i has side information $\{X_j : j \in S_i\}$.
- User i requests packet X_i.
- $\mathcal{I} = \{S_i : i \in [n]\}$ is an instance of the index coding with side-information problem.

Problem 1 (The Main ICSI Problem)

What is the minimum number of transmissions required by the sender to satisfy all n requests, if encoding of data is permitted?

- Related Structure: the side-information digraph

Index Coding with Side Information



Example 2

Sender has $\{X_1, X_2, X_3, X_4\}, X_i \in \mathbb{F}_2^t$. The receivers have side-information:

$$D_1 = \{X_2, X_3, X_4\}, D_2 = \{X_1, X_3, X_4\},$$

$$D_3 = \{X_1, X_2, X_4\}, D_4 = \{X_1, X_2, X_3\}.$$

Choosing $L = [1, 1, 1, 1]$ the sender broadcasts LX :

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \longrightarrow LX = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = X_1 + X_2 + X_3 + X_4$$

So the minimum number of transmissions is $N = 1$.

The Min-Rank of a Graph

Definition 3

Let G be a directed graph with adjacency matrix $\mathcal{A}$.

$$\mathsf{minrank}_q(G) := \min\{\mathrm{rank}_q(A + I) : \mathrm{Supp}(A) \subset \mathrm{Supp}(\mathcal{A})\}.$$

Theorem 4 (Bar-Yossef, Birk, Jayram, Kol 2006)

The minimum number of transmissions required for a linear index code over $\mathbb{F}_2$ for the instance $\mathcal{I}$ is minrank(G), where G is the side-information graph of $\mathcal{I}$.

- The minrank is NP-hard to compute (Peeters, 1996)

Bounds on the Min-Rank of a Graph

Theorem 5 (Haemers, Haviv & Langberg, Bar-Yossef *et al*)

For every undirected graph G of n vertices over $\mathbb{F}_q$:

- $\alpha(G) \leq \Theta(G) \leq minrank_q(G) \leq \chi(\overline{G})$
- $\Omega(\log n) \leq minrank_q(G(n,p)) \leq \mathcal{O}(n/\log n)$
- *Expected value of $minrank_q(G(n,p))$ is (almost surely) $\Omega(\sqrt{n})$*

- $\alpha(G)$ is the max size of an independent set
- $\Theta(G)$ is the Shannon capacity of G
- $\chi(\overline{G})$ is the chromatic number of $\overline{G}$

Graph Theory

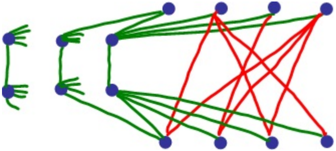
Shanmugam, Dimakis, Langberg, "Graph Theory Versus Minimum Rank for Index Coding," (2014) arXiv.1402.3898
The authors:

- distinguish between 'graph theoretic' and 'algebraic' methods,
- give index coding schemes from graph theory that outperform all known graph theoretic bounds,
- show all known graph theoretic bounds are withing $\log n$ of the chromatic number,
- state that the minrank (algebraic) can outperform the chromatic number by a polynomial factor.

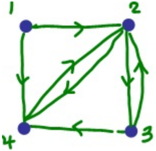
Equivalence of Linear Network and Index Coding

Theorem 6 (El Rouyhab et al 2010)

There exists a linear network code if and only if there exists a perfect linear index code.



Network Code



Index Code

Index Coding with Side-Information (New Formula)

- The sender has a file X split into n packets $X = [X_1, ..., X_n] \in \mathbb{F}_q^n$.
- There are $m \geq n$ users $\{1, ..., m\}$.
- User i has side information $\{X_j : j \in S_i\}$.
- User i requests packet $X_{f(i)}$, some surjection $f : [m] \longrightarrow [n]$.

Problem 7 (The Main ICSI Problem)

What is the minimum number of transmissions required by the sender to satisfy all m requests, if encoding of data is permitted?

- Related Structure: the side-information hypergraph

Data Retrieval

Definition 8

We say that $L \in \mathbb{F}_q^{N\times n}$ represents an linear $\mathcal{I} = (n, m, S, f)$ of the index coding problem with side information indexed by $S = \{S_i : i \in [m]\}$ if for each receiver $i \in [m]$ there is a decoding map

$$D_i : \mathbb{F}_q^N \times \mathbb{F}_q^n \to \mathbb{F}_q,$$

such that for some $A \in \mathbb{F}_q^n, \mathrm{Supp}(A) \subset S_i$

$$D_i(LX, A) = X_{f(i)} \ \forall X \in \mathbb{F}_q^n.$$

Decoding at the Receiver i

Let $A \in \mathbb{F}_q^n$ s.t. $\mathrm{Supp}(A) \subset S_i$. User i knows $AX = \sum_{j \in S_i} A_j X_j$.
Let $B \in \mathbb{F}_q^N$ such that

$$BL = A + e_{f(i)}. \tag{1}$$

Then

$$BLX = AX + e_{f(i)}X = AX + X_{f(i)}.$$

So the existence of a decoder depends on the solvability of (1).

$$D_i(LX, A) = BLX - AX = X_{f(i)}$$

The Min-Rank

Theorem 9 (Dau, Skachek, Chee 2012)

The minimum number of transmissions required for an instance $\mathcal{I} = (n, m, S, f)$ of the index coding problem is

$$\kappa(\mathcal{I}) := \min\{\mathrm{rank}(U + E_f) : \mathrm{Supp}(U_i) \subset S_i, i \in [m]\},$$

where $E_f \in \mathbb{F}_q^{m\times n}$ has each ith row equal to $e_{f(i)}$.

- $\kappa(\mathcal{I})$ is called the minrank of the system.
- $\kappa(\mathcal{I})$ generalizes the minrank of the *side-information graph*
- $\kappa(\mathcal{I})$ is *NP*-hard to compute.

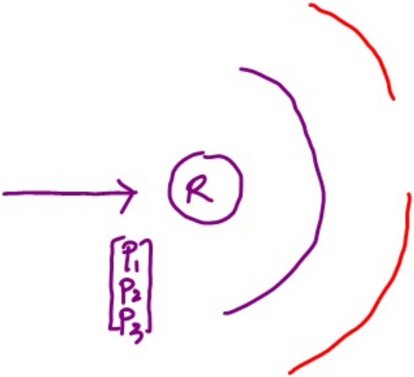
Coded-Side Information

User i wants P_i. The sender transmits a packet at each time slot.

Slot	Sent	User 1?	User 2?	User 3?
1	P_1	N	Y	N
2	P_2	Y	N	N
3	P_3	Y	Y	N
4	$P_1 + P_2$	N	N	Y
5	$P_1 + P_2 + P_3$	Y	Y	Y

After 5 transmissions, all user requests have been satisfied.

Coded-Side Information



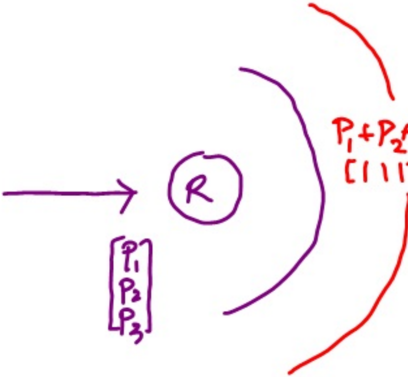
$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{bmatrix} P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{bmatrix} P_1 + P_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Coded-Side Information



$$\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$$

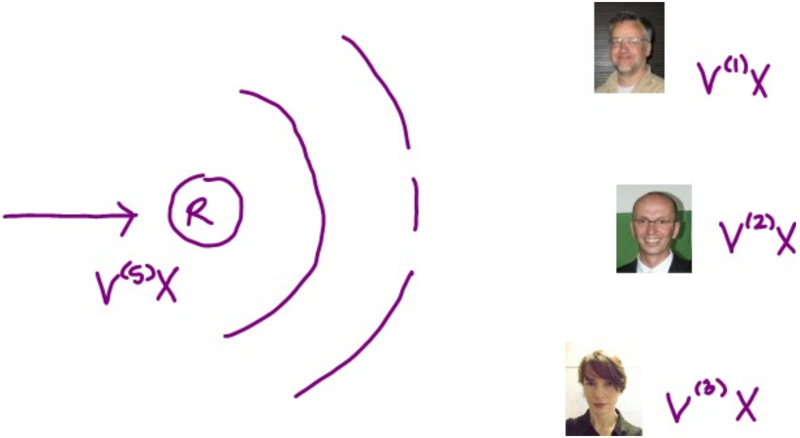
$$P_1+P_2+P_3 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$$

$$\begin{bmatrix}P_2\\P_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$$

$$\begin{bmatrix}P_1\\P_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$$

$$\begin{bmatrix}P_1+P_2\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}$$

Coded-Side Information



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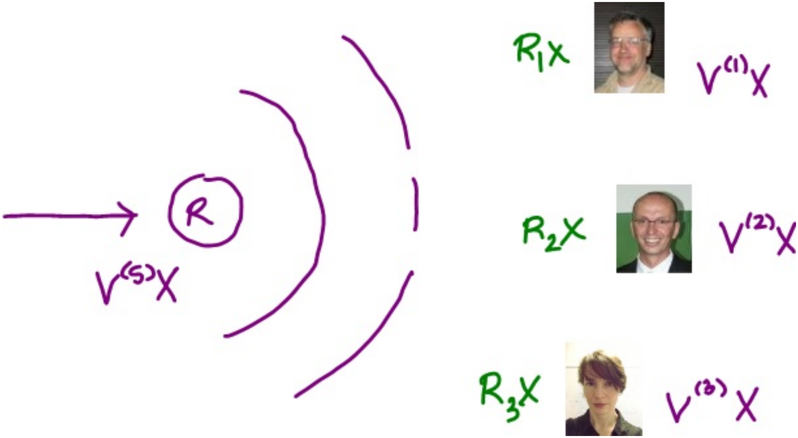
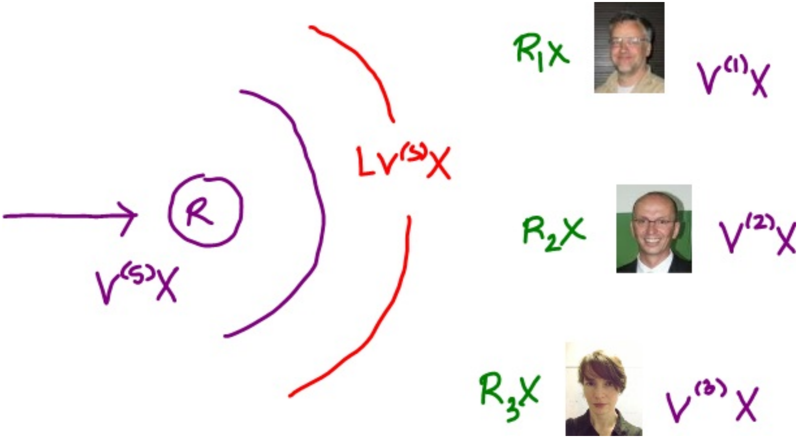


Coded-Side Information



Index-Coding with Coded-Side Information (New 3-in-1)

- $X \in \mathbb{F}_q^{n \times t}$
- The sender has $V^S X \in \mathbb{F}_q^{d_S \times t}$
- User i wants the packet $R_i X \in \mathbb{F}_q^t$,
- User i has side information $(V^{(i)}, V^{(i)} X) \in \mathbb{F}_q^{d_i \times n} \times \mathbb{F}_q^{d_i \times t}$
- The sender transmits $Y = LV^S X \in \mathbb{F}_q^{N \times t}$, some $L \in \mathbb{F}_q^{N \times d_S}$

Objective 1

The sender aims to find an encoding $LV^{(S)}X$ that minimizes N such that the demands of all users satisfied.

Case $t = 1$: Shum, Mingjun, Sung, "Broadcasting with Coded Side Information", IEEE 23rd PIMRC, vol. 89, no. 94, pp. 9-12, 2012.

An Instance of the ICCSI Problem

Definition 10

An instance of the Index Coding with Coded Side Information (ICCSI) problem is a list

$$\mathcal{I} = (t, n, m, \mathcal{X}, \mathcal{X}^S, R),$$

satisfying:

- t, n, m are positive integers,
- $\mathcal{X} = \oplus_{i \in [m]} \mathcal{X}^{(i)}$,
- $\mathcal{X}^{(i)} := \langle V^{(i)} \rangle < \mathbb{F}_q^n$, $\mathrm{dim} \mathcal{X}^{(i)} = d_i$,
- $\mathcal{X}^S := \langle V^{(S)} \rangle < \mathbb{F}_q^n$, $\mathrm{dim} \mathcal{X}^S = d_S$,
- $R \in \mathbb{F}_q^{m \times n}$ has rows $R_i \in \mathbb{F}_q^n$,
- $R_i \in \mathcal{X}^S$, $i \in [m]$.

Linear Index Encoding

Definition 11

Let N be a positive integer. The map

$$E : \mathbb{F}_q^{n \times t} \to \mathbb{F}_q^{N \times t},$$

is an $\mathbb{F}_q$-index code for $\mathcal{I}$ (E is an $\mathcal{I}$-IC) of length N if for each $i \in [m]$ there exists a decoding map

$$D_i : \mathbb{F}_q^{N \times t} \times \mathcal{X}^{(i)} \to \mathbb{F}_q^t,$$

satisfying

$$\forall X \in \mathbb{F}_q^{n \times t} \ : \ D_i(E(X), A) = R_i X,$$

for some $A \in \mathcal{X}^{(i)}$. E is called an $\mathbb{F}_q$-linear $\mathcal{I}$-IC if $E(X) = LV^{(S)}X$ for some $L \in \mathbb{F}_q^{N \times d_S}$. Then L represents the $\mathcal{I}$-IC E.

Decoding Criteria

Lemma 12

$L \in \mathbb{F}_q^{N \times d_s}$ represents an $\mathcal{I}$-IC if and only if for each $i \in [m]$,

$$R_i \in \left\langle \begin{bmatrix} V^{(i)} \\ LV^{(S)} \end{bmatrix} \right\rangle .$$

User i can compute

$$R_i X = AV^{(i)} X + BLV^{(S)} X,$$

for any $A \in \mathbb{F}_q^{d_i}$, $B \in \mathbb{F}_q^{N}$ satisfying $R_i = AV^{(i)} + BLV^{(S)}$.

The Min-Rank

Lemma 13 (BC)

The length of an optimal $\mathbb{F}_q$-linear $\mathcal{I}$-IC is $\kappa(\mathcal{I}) :=$

$$\min\{\mathrm{rank}(A + R) : A \in \mathbb{F}_q^{m \times n}, A_i \in \mathcal{X}^{(i)} \cap \mathcal{X}^S < \mathbb{F}_q^n, \ \forall i \in [m]\}.$$

- $\kappa(\mathcal{I})$ is called the minrank of the instance $\mathcal{I}$.
- $\kappa(\mathcal{I}) = d_{\mathrm{rk}}(R, \mathcal{X} \cap \tilde{\mathcal{X}}) = w_{\mathrm{rk}}(R + (\mathcal{X} \cap \tilde{\mathcal{X}}))$, where $\tilde{\mathcal{X}} = \oplus \mathcal{X}^S$.

The Min-Rank

$m = 6, n = 4, R_i = e_i, i \in [m], R_5 = e_2, R_6 = e_1$ over $\mathbb{F}_2$.

$$V^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, V^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, V^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$V^{(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, V^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, V^{(6)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R + \mathcal{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & * & * \\ * & 0 & 0 & * \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{bmatrix} = \begin{bmatrix} 1 & 0 & * & * \\ * & 1 & 0 & * \\ * & * & 1 & 0 \\ 0 & * & * & 1 \\ * & 1 & * & 0 \\ 1 & * & 0 & * \end{bmatrix}$$

The minrank is 3, so $N = 3$ transmissions are required.

Existence of an $\mathcal{I}$-IC

Theorem 14 (BC)

Let $\mathcal{I}$ be an instance of an ICCSI problem and let

$$N = \max\{n - d_i : i \in [m]\}.$$

Suppose that $q > m$. If L is chosen uniformly at random in $\mathbb{F}_q^{N \times d_S}$ then the probability that L represents a linear $\mathcal{I}$-IC is at least $(1 - m/q)^N$.

Corollary 15

If $q > m$ then $\kappa(\mathcal{I}) \leq \max\{n - d_i : i \in [m]\}$.

- Comparable with the Main Network Coding Theorem (see Fragouli & Soljanin *Network Coding Fundamentals*).

The Min-Rank

$m = 6, n = 4, R_i = e_i, i \in [m], R_5 = e_2, R_6 = e_1$ over $\mathbb{F}_3$.

$$V^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, V^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, V^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$V^{(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, V^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, V^{(6)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R + \mathcal{X} = \begin{bmatrix} 1 & 0 & * & * \\ * & 1 & 0 & * \\ * & * & 1 & 0 \\ 0 & * & * & 1 \\ * & 1 & * & 0 \\ 1 & * & 0 & * \end{bmatrix} \text{ contains } \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

$R + \mathcal{X}$ has minrank $2 = N = n - d_i$ over $\mathbb{F}_3$.

Bounds on the Min-Rank

- Can we improve the bound $\kappa(\mathcal{I}) \leq \max\{n - d_i : i \in [m]\},\ q > m$?
- An $\mathcal{I}$-IC of length N exists if and only if, for all $i \in [m]$,

$$(R_i + \mathcal{X}^{(i)}) \cap \mathcal{L} \neq \emptyset$$

where $\mathcal{L} = \langle LV^{(S)} \rangle$.

Problem 16

Find an upper bound on the number of N-dimensional subspaces $\mathcal{L}$ that miss $R_i + \mathcal{X}^{(i)}$ for at least one value of $i \in [m]$.

Bounds on the Min-Rank

Using incidence matrices of designs we can construct $\mathcal{I}$ and compute bounds on $\kappa(\mathcal{I})$.

Theorem 17

(BC) There exist $\mathcal{I}$-IC satisfying

- $\kappa(\mathcal{I}) \leq \frac{m+1}{2}$ *(2-(n, k, λ) design),*
- $\kappa(\mathcal{I}) = \frac{p^2+p+1}{2}$ *(projective plane of order $p = 2, 3$).*

Index Coding with Errors

Dau, Skachek, Chee, "Error Correction for Index Coding With Side Information," IEEE Trans on Inform. Th, (59), 3, 2013.

The authors:

- introduced index coding with error correction,
- gave several bounds on the length of an optimal δ-error index code,
- gave a decoding algorithm based on syndrome decoding.

Error-Correction for Coded-Side Information

Definition 18

Let $\mathcal{M} \subset \mathbb{F}_q^{n\times t}$ be the message space.

$$E : \mathbb{F}_q^{n\times t} \to \mathbb{F}_q^{N\times t},$$

is a δ-error correcting code for $\mathcal{I}$ of length N (E is $(\mathcal{I},\delta)$-ECIC), if for each i $\exists$ a decoding map

$$D_i : \mathbb{F}_q^{N\times t} \times \mathcal{X}^{(i)} \to \mathbb{F}_q^{t},$$

such that for some $A \in \mathcal{X}^{(i)}$.

$$D_i(E(X) + W, A) = R_i X$$

for all $X \in \mathcal{M}$ and $W \in \mathbb{F}_q^{N\times t}$, $w(W) \leq \delta$.
E is linear if $E(X) = LV^{(S)}X$ for some $L \in \mathbb{F}_q^{N\times d_S}$.

Decoding Criterion

Theorem 19 (BC)

Let $\mathcal{I}$ be an instance of an ICCSI problem and let N be a positive integer. A matrix $L \in \mathbb{F}_q^{N \times d_S}$ represents a linear $(\mathcal{I}, \delta)$-ECIC if and only if for all $i \in [m]$

$$w\left(LV^{(S)}(X - X')\right) \geq 2\delta + 1,$$

for all $X, X' \in \mathcal{M}$ such that $X - X' \in \mathcal{Z}^{(i)}$.

- $\mathcal{Z}^{(i)} = \{Z \in \mathbb{F}_q^{n \times t} : V^{(i)}X = 0, R_i X \neq 0\}$

Bounds on the Optimal Length of an ECIC: $t = 1$

- $\mathcal{N}(\mathcal{I}, \delta) =$ optimal length N of an δ-error correcting $\mathcal{I}$.
- $N(k, d) =$ optimal length ℓ of a $\mathbb{F}_q$-$[\ell, k, d]$ code
- $\mathcal{J}(\mathcal{I}) := \{U < \mathbb{F}_q^n : U \backslash \{0\} \subset \cup_{i \in [m]} \mathcal{Z}^{(i)}\}$.
- $\alpha(\mathcal{I}) := \max\{\dim U : U \in \mathcal{J}(\mathcal{I})\}$
- $\alpha(\mathcal{I})$ generalizes the notion of an independent set.

Theorem 20 (BC)

Let $\mathcal{I}$ be an instance of the ICCSI problem with $t = 1$. Then

- $N(\alpha(\mathcal{I}), 2\delta + 1) \leq \mathcal{N}(\mathcal{I}, \delta)$,
- $\alpha(\mathcal{I}) \leq \kappa(\mathcal{I})$.

Further Bounds..

Theorem 21 (BC)

Let $\mathcal{I}$ be an instance of the ICCSI problem with $t = 1$. Then

- $\mathcal{N}(\mathcal{I}, \delta) \leq N(\kappa(\mathcal{I}), 2\delta + 1)$ *(κ-bound),*
- $\kappa_q(\mathcal{I}) + 2\delta \leq \mathcal{N}_q(\mathcal{I}, \delta)$ *(Singleton Bound),*
- *if* $q \geq \kappa(\mathcal{I}) + 2\delta - 1$ *then* $\mathcal{N}(\mathcal{I}, \delta) = \kappa(\mathcal{I}) + 2\delta$*,*
- *there exists an* $\mathbb{F}_q$*-linear* $(\mathcal{I}, \delta)$*-ECIC if*

$$N > n - d - 1 + \log_q(m(q-1)V_q(N, 2\delta)),$$

where $d = \min\{d_i : i \in [m]\}$*.*

Bounds on the Optimal Length of an ECIC: $t > 1$

- $\mathcal{N}(\mathcal{I}, \delta) =$ optimal length N of an δ-error correcting $\mathcal{I}$.
- $N(t, \log_q M, d)$ is the least integer N s.t. $\exists$ a code in $\mathbb{F}_q^{N \times t}$ of minimum rank distance d and size M.
- $\mathcal{J}(\mathcal{I}) := \{U \subset \mathbb{F}_q^{n \times t} : X - X' \in \mathcal{Z}_\delta^{(i)}$ some i, any $X, X' \in U\}$.
- $\alpha(\mathcal{I}) := \max\{\log_q |U| : U \in \mathcal{J}(\mathcal{I})\}$
- $\alpha(\mathcal{I})$ generalizes the notion of an independent set.

Theorem 22 (BC)

Let $\mathcal{I}$ be an instance of the ICCSI problem with $t > 1$. Then

- $N(\alpha(\mathcal{I}), 2\delta + 1) \leq \mathcal{N}(\mathcal{I}, \delta)$,
- $\mathcal{N}(\mathcal{I}, \delta) \geq \frac{\alpha(\mathcal{I})}{t} + 2\delta$ *if* $t \geq N(t, \alpha(\mathcal{I}), 2\delta + 1)$,
- $\mathcal{N}(\mathcal{I}, \delta) \geq \frac{\alpha(\mathcal{I})}{t - 2\delta}$ *if* $t \leq N(t, \alpha(\mathcal{I}), 2\delta + 1)$,
- $\alpha(\mathcal{I}) \leq \kappa(\mathcal{I})$.

Decoding

- For Hamming errors a variation of syndrome decoding can be used (high complexity).
- Adding further redundancy to the system, we can use a simple matrix decoder (Silva et al 2010) to correct rank-metric errors in linear time.

Decoding Over the Matrix Channel

The sender transmits:

$$A = \begin{pmatrix} 0_{\delta\times\delta} & 0_{\delta\times t} \\ 0_{N\times\delta} & LV^{(S)}X \end{pmatrix},$$

The error matrix has the form

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

and $\mathrm{rank}(W_{11}) = \mathrm{rank}(W) = r \leq \delta$. The receivers get

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} + LV^{(S)}X \end{pmatrix} - - > \begin{pmatrix} W_{11} & W_{12} \\ 0 & LV^{(S)}X \end{pmatrix}.$$

Decoding

1. Choose $A \in \mathbb{F}_q^{d_i}$.
2. Solve $R_i + AV^{(i)} = BLV^S$ for some $B \in \mathbb{F}_q^N$.
3. Compute $R_i X = BY - AV^{(i)} X$.

In other words, the decoder computes $M = [G|H]$, the reduced-row echelon form of the matrix

$$\begin{bmatrix} V^{(i)} & V^{(i)}X \\ LV^{(S)} & Y \end{bmatrix}$$

and solves for Z in $ZA = R_i$ to retrieve $R_i X = ZB$.

Side-Information Coverings

Lemma 23 (BC)

Given $t, n, m, \mathcal{X}$, there exists an encoding of length N satisfying every possible request R if and only if

$$\begin{aligned} N &\geq \max\{\min\{\mathrm{rank}(U+R) : U \in \mathcal{X}\}, R \in \mathbb{F}_q^{m\times n}\} \\ &= \max\{\min\{d_{\mathrm{rk}}(R,U) : U \in \mathcal{X}\}, R \in \mathbb{F}_q^{m\times n}\} \\ &= \max\{d_{\mathrm{rk}}(R,\mathcal{X}), R \in \mathbb{F}_q^{m\times n}\} \\ &= \rho_{\mathrm{rk}}(\mathcal{X}) \end{aligned}$$

$=$ the rank-metric covering radius of $\mathcal{X} = \mathcal{X}^{(1)} \oplus \cdots \oplus \mathcal{X}^{(m)}$.

The Caching Problem

M. A. Maddah-Ali, U. Niesen, "Fundamental Limits of Caching," arXiv.1209.5807.

- m users seek all or part of a file $X \in \mathbb{F}_q^{n\times t}$
- Each user has storage capacity d_i.
- Placement phase: the sender places data in each user's cache during low-traffic times.
- Delivery phase: the sender broadcasts data according to users demands.
- User requests are unknown to the sender at the placement phase.

Problem 24

How should data files be placed in order to minimize transmissions during delivery?

A Caching Strategy

- There are m users each with storage capacity d_i, $i \in [m]$.
- Choose $\mathcal{X} = \oplus_{i\in[m]}\mathcal{X}^{(i)}$ s.t.
  - $\dim \mathcal{X}^{(i)} = d_i$
  - $\mathcal{X}$ has optimal covering rank radius N
- Place $\mathcal{X}^{(i)}$ in User i's cache.
- Then at the delivery phase all possible users' requests can be satisfied with N transmissions.

Problem 25

Construct codes $\mathcal{X}$ with low covering radius.

Final Remarks

- There is an analogue of the ICSI problem for subspace codes. Each user i has some side information $\mathcal{X}^{(i)} < \mathbb{F}_q^n$ and the sender transmits $V < \mathrm{rowsp}(X)$ such that a requested 1-dimensional subspace of $\mathrm{rowsp}(X)$ is contained in $V + \mathcal{X}^{(i)}$.
- How should the ICSI problem be modelled for implementation with MRD codes?
- What bounds and caching schemes in the caching problem can be obtained via algebraic codes for the rank metric?