Distance regular colorings of Cayley graphs of $\mathbb{Z}^n$

Anastasia Vasil'eva

Sobolev Institute of Mathematics,
Novosibirsk State University,
Novosibirsk, RUSSIA

Perfect colorings

Let $G = (V(G), E(G))$ be a graph.
We denote the graph distance between $\alpha, \beta \in V(G)$ by $d(\alpha, \beta)$.
Let $\overline{C} = (C_1, \ldots, C_r)$ be a partition (or a coloring) of $V(G)$.
We identify the partition $\overline{C} = (C_1, \ldots, C_r)$ with the function $\varphi : V(G) \to \{1, \ldots, r\}$.

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Suppose that for any $i, j \in \{1, \ldots, r\}$ there exists the integer a_{ij} such that for any $\alpha \in C_i \quad |\{\beta \in C_j \ : \ d(\alpha, \beta) = 1\}| = a_{ij}$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix}$$

Then the coloring $\overline{C}$ is called perfect (or the partition is called equitable) with the parameter matrix A.

Distance regular colorings

If the parameter matrix is three-diagonal then the coloring is called distance regular. In this case we denote

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$$A = \begin{bmatrix} k_1 & u_1 & 0 & & & 0 \\ l_2 & k_2 & u_2 & & & \\ & \ddots & \ddots & \ddots & & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & l_{r-1} & k_{r-1} & u_{r-1} \\ 0 & & & 0 & l_r & k_r \end{bmatrix}$$

In this case the code C_1 (and C_r) is called completely regular.

Cayley graphs

We consider the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators $\{g_1, \dots, g_m\} \in \mathbb{Z}^n$. Here
$V(Cay(\mathbb{Z}^n, m)) = \mathbb{Z}^n$ and
$(\alpha, \beta) \in E(Cay(\mathbb{Z}^n, m)) \iff \alpha - \beta = \pm g_i$ for some i.

Distance regular graphs

Usually perfect colorings (equitable partitions) and completely regular codes are considered in distance regular graphs i.e.; the graphs with the following property:
for any $\alpha \in V(G)$ the distance partition $(C_1 = \{\alpha\}, \dots, C_r)$ is distance regular with the same parameters.

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Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators are not distance regular for any $n \geq 2$.

Monotonicity

Let C be a completely regular code and $\overline{C} = (C_1 = C, \ldots, C_r)$ be a distance regular coloring. In general, even in an arbitrary distance regular graph the sequences $(l_2, \ldots, l_r)$ and $(u_1, \ldots, u_{r-1})$ are not monotonic.

Theorem 1.

Let $\overline{C} = (C_1, \ldots, C_r)$ be a distance regular coloring of the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators. Then

$$l_2 \leq \ldots \leq l_r,$$

$$u_1 \geq \ldots \geq u_{r-1}.$$

Reducible colorings

We identify the partition $\overline{C} = (C_1, \ldots, C_r)$ with the function $\varphi : V(G) \to \{1, \ldots, r\}$. The coloring φ of the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators $\{g_1, \ldots, g_m\}$ is called reducible if it can represented as follows:
there exists the distance regular coloring ψ of the Cayley graph of $\mathbb{Z}^1$ and the constants $c_1, \ldots, c_m \in \{-1, 0, 1\}$ such that for any
$\alpha = z_1 g_1 + \ldots + z_m g_m \in \mathbb{Z}^n$
$\varphi(\alpha) = \psi(z_1 c_1 + \ldots + z_m c_m)$

The number of colors

Theorem 2.

Let $\overline{C} = (C_1, \dots, C_r)$ be an irreducible distance regular coloring of the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators. Then $r \leq 2m + 1$.

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Or in terms of codes:

Theorem 2'.

Let C be a completely regular code in the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators. Then the covering radius of C is at most $2m + 1$.

The number of colors

There is the homomorphism h of the graph $C(\mathbb{Z}^m)$ of m-dimensional rectangular grid into the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators $\{g_1, \dots, g_m\}$:
$h(\alpha_1, \dots, \alpha_m) = \alpha_1 g_1 + \dots + \alpha_m g_m$.
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So, any coloring of $Cay(\mathbb{Z}^n, m)$ produces the coloring of $C(\mathbb{Z}^m)$.
There exists the unique distance regular coloring of $C(\mathbb{Z}^m)$ with $2m+1$ colors. This coloring is generated by the distance coloring of $\mathbb{Z}_2^{2m} = \mathbb{Z}_4^m$ with respect to one vertex. It can not generate the distance regular coloring with $2m+1$ colors in $Cay(\mathbb{Z}^n, m)$.

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Theorem 3.

Let $\overline{C} = (C_1, \ldots, C_r)$ be an irreducible distance regular coloring of the Cayley graph $Cay(\mathbb{Z}^n, m)$ of $\mathbb{Z}^n$ with m generators. Then $r \leq 2m$.

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Thank you for your attention!