Construction of simple 3-designs using resolution

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OUTLINE

- Generic constructions
- Applications
- $(1, \sigma)$-resolvable 3-designs

Generic constructions

Definition

A $t-(v,k,\lambda)$*-design* $(X,\mathcal{B})$ *is said to be* (s,σ)*-resolvable if its block set* $\mathcal{B}$ *can be partitioned into w classes* $\pi_1,\ldots,\pi_w$ *such that* (X,π_i) *is a* $s-(v,k,\sigma)$ *design for all* $i=1,\ldots,w$*, where* $1\leqslant s\leqslant t$*. Each* π_i *is called a resolution class*

GENERIC CONSTRUCTIONS

Definition

A $t-(v,k,\lambda)$-design $(X,\mathcal{B})$ is said to be (s,σ)-resolvable if its block set $\mathcal{B}$ can be partitioned into w classes $\pi_1,\ldots,\pi_w$ such that (X,π_i) is a $s-(v,k,\sigma)$ design for all $i=1,\ldots,w$, where $1\leqslant s\leqslant t$. Each π_i is called a resolution class

Definition

Let D be a $t-(v,k,\lambda)$ design (D may have repeated blocks) admitting a (s,σ)-resolution with $\pi_1,\ldots,\pi_w$ as resolution classes. Define a distance between any two classes π_i and π_j by $d(\pi_i,\pi_j)=\min\{|i-j|,w-|i-j|\}$.

GENERIC CONSTRUCTIONS

- $n \geqslant 1$, integer.
- $\{k_1, \ldots, k_n, k_{n+1}, \ldots, k_{2n}\}$ and k, integers, such that $2 \leqslant k_1 < \ldots < k_n < k/2$ and $k_i + k_{n+i} = k$ for $i = 1, \ldots, n$.

Generic constructions

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- Assume there exist $2n$ 3-designs $D_i = (X, \mathcal{B}_i)$ with parameters $3 - (v, k_i, \lambda^{(i)})$ having a $(1, \sigma^{(i)})$-resolution such that $w_i = w_{n+i}$ for all $i = 1, \ldots, n$, where w_j is the number of $(1, \sigma^{(j)})$-resolution classes of D_j.

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- Assume there exist $2n$ 3-designs $D_i = (X, \mathcal{B}_i)$ with parameters $3-(v, k_i, \lambda^{(i)})$ having a $(1, \sigma^{(i)})$-resolution such that $w_i = w_{n+i}$ for all $i = 1, \ldots, n$, where w_j is the number of $(1, \sigma^{(j)})$-resolution classes of D_j.

- Also assume that
  1. For each pair (D_i, D_{n+i}), $1 \leqslant i \leqslant n$, either D_i or D_{n+i} has to be simple.

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- $\{k_1, \ldots, k_n, k_{n+1}, \ldots, k_{2n}\}$ and k, integers, such that $2 \leqslant k_1 < \ldots < k_n < k/2$ and $k_i + k_{n+i} = k$ for $i = 1, \ldots, n$.
- Assume there exist $2n$ 3-designs $D_i = (X, \mathcal{B}_i)$ with parameters $3-(v, k_i, \lambda^{(i)})$ having a $(1, \sigma^{(i)})$-resolution such that $w_i = w_{n+i}$ for all $i = 1, \ldots, n$, where w_j is the number of $(1, \sigma^{(j)})$-resolution classes of D_j.
- Also assume that
  1. For each pair (D_i, D_{n+i}), $1 \leqslant i \leqslant n$, either D_i or D_{n+i} has to be simple.
  2. If a D_j, $j \in \{i, n+i\}$, is not simple, then D_j is a union of a_j copies of a simple $3-(v, k_j, \alpha^{(j)})$ design C_j, wherein C_j admits a $(1, \sigma^{(j)})$-resolution. Thus, $\lambda^{(j)} = a_j \alpha^{(j)}$.

Generic constructions

- If D_j is not simple, (i.e. D_j is a union of a_j copies of a simple $3-(v, k_j, \alpha^{(j)})$ design C_j, where $P^{(j)} = \{\pi_1^{(j)}, \ldots, \pi_{t_j}^{(j)}\}$ is a $(1, \sigma^{(j)})$-resolution of C_j), then the corresponding $(1, \sigma^{(j)})$-resolution of D_j is the concatenation of a_j sets $P^{(j)}$. So, the $w_j = a_j t_j$ resolution classes of D_j are of the form

$$\pi_1^{(j)}, \ldots, \pi_{t_j}^{(j)}, \quad \pi_1^{(j)}, \ldots, \pi_{t_j}^{(j)}, \quad \ldots, \quad \pi_1^{(j)}, \ldots, \pi_{t_j}^{(j)}$$

Generic constructions

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$$\pi_1^{(j)},\ldots,\pi_{t_j}^{(j)},\quad \pi_1^{(j)},\ldots,\pi_{t_j}^{(j)},\quad \ldots,\quad \pi_1^{(j)},\ldots,\pi_{t_j}^{(j)}$$

- If $k_1=2$, then D_1 is a union of a_1 copies of the trivial $2-(v,2,1)$ design i.e. D_1 is considered as a 3-design with $\lambda^{(1)}=0$.

Generic constructions

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- If $k_1=2$, then D_1 is a union of a_1 copies of the trivial $2-(v,2,1)$ design i.e. D_1 is considered as a 3-design with $\lambda^{(1)}=0$.

- If necessary, also assume that there exists a $3-(v,k,\Lambda)$ design $D=(X,\mathcal{B})$.

Generic constructions

Notation:

- $\pi_1^{(\ell)}, \ldots, \pi_{w_\ell}^{(\ell)}$: the w_ℓ classes in a $(1, \sigma^{(\ell)})$-resolution of D_ℓ, $\ell = 1, \ldots, 2n$. Recall that $w_{n+h} = w_h$.
- The distance defined on the classes of D_ℓ is then $d^{(\ell)}(\pi_i^{(\ell)}, \pi_j^{(\ell)}) = \min\{|i-j|, w_\ell - |i-j|\}$.
- $b^{(j)} = \sigma^{(j)} v/k$: the number of blocks in each resolution class of of D_j.
- $u_j = \sigma^{(j)}$: the number of blocks containing a point in each resolution class of of D_j.
- $\lambda_2^{(j)} = \lambda^{(j)}(v-2)/(k_j - 2)$: the number of blocks of D_j containing two points.

Construction I

Let $\tilde{D}_i = (\tilde{X}, \tilde{\mathcal{B}}_i)$ be a copy of D_i defined on $\tilde{X}$ such that $X \cap \tilde{X} = \varnothing$. Also let $\tilde{D} = (\tilde{X}, \tilde{\mathcal{B}})$ be a copy of D.

Define blocks on the point set $X \cup \tilde{X}$ as follows:

I. blocks of D and blocks of $\tilde{D}$;

II. blocks of the form $A \cup \tilde{B}$ for any pair $A \in \pi_i^{(h)}$ and $\tilde{B} \in \tilde{\pi}_j^{(n+h)}$ with $\varepsilon_h \leqslant d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leqslant s_h$, $\varepsilon_h = 0, 1$, for $h = 1, \ldots, n$;

III. blocks of the form $\tilde{A} \cup B$ for any pair $\tilde{A} \in \tilde{\pi}_i^{(h)}$ and $B \in \pi_j^{(n+h)}$ with $\varepsilon_h \leqslant d^{(h)}(\pi_i^{(h)}, \pi_j^{(h)}) \leqslant s_h$, $\varepsilon_h = 0, 1$, for $h = 1, \ldots, n$.

Denote $z_h := (2s_h + 1 - \varepsilon_h)$ for $h = 1, \ldots, n$.

Generic constructions

Verification: CASE $k_i \geqslant 3$

- The blocks containing points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$):

$$\Lambda + \sum_{h=1}^{n} z_h \lambda^{(h)} b^{(n+h)} + z_h \lambda^{(n+h)} b^{(h)}$$

- The blocks containing points $a, b, \tilde{c}$ with $a, b \in X$ and $\tilde{c}, \in \tilde{X}$ (resp. $\tilde{a}, \tilde{b}, c$):

$$\sum_{h=1}^{n} z_h \lambda_2^{(h)} u_{n+h} + z_h \lambda_2^{(n+h)} u_h$$

- The defined blocks will form a 3-design if

$$\Lambda + \sum_{h=1}^{n} z_h \lambda^{(h)} b^{(n+h)} + z_h \lambda^{(n+h)} b^{(h)} = \sum_{h=1}^{n} z_h \lambda_2^{(h)} u_{n+h} + z_h \lambda_2^{(n+h)} u_h,$$

equivalently

$$\Lambda = \sum_{h=1}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h.$$

Generic constructions

Verification: CASE $K_1 = 2$

The condition for which the defined blocks form a 3-designs becomes

$$\begin{aligned}\Lambda &= \{a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}\} z_1 \\ &+ \sum_{h=2}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h.\end{aligned}$$

Summary of Construction I

(i) If $k_1 = 2$ and

$$0 = \{a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}\} z_1 + \sum_{h=2}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (1)$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, then there exists a $3-(2v, k, \Theta)$ design with

$$\Theta = \{a_1 u_{n+1} + \lambda_2^{(n+1)} u_1\} z_1 + \sum_{h=2}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

(ii) If $k_1 \geqslant 3$ and

$$0 = \sum_{h=1}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (2)$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, then there exists a $3-(2v, k, \Theta)$ design with

$$\Theta = \sum_{h=1}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

Summary of Construction I (Cont.)

(iii) If $k_1 = 2$ and

$$\begin{aligned} 0 &< \{a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}\} z_1 \\ &+ \sum_{h=2}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (3) \end{aligned}$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, further if there is a $3-(v,k,\Lambda)$ design having

$$\begin{aligned} \Lambda &= \{a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}\} z_1 \\ &+ \sum_{h=2}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (4) \end{aligned}$$

then there exists a $3-(2v,k,\Theta)$ design with

$$\Theta = \{a_1 u_{n+1} + \lambda_2^{(n+1)} u_1\} z_1 + \sum_{h=2}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

Summary of Construction I (Cont.)

(iv) If $k_1 \geqslant 3$ and

$$0 < \sum_{h=1}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (5)$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, further if there is a $3-(v,k,\Lambda)$ design having

$$\Lambda = \sum_{h=1}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (6)$$

then there exists a $3-(2v,k,\Theta)$ design with

$$\Theta = \sum_{h=1}^{n} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

Construction II

Construction II deals with the case $k_n = k/2$.

Take $D_n = D_{2n}$.

Blocks of types I, II, and III are as in Construction I for $h = 1, \ldots, n-1$. Define a further type of blocks.

IV. blocks of the form $A \cup \tilde{B}$ for any pair $A \in \pi_i^{(n)}$ and $\tilde{B} \in \tilde{\pi}_j^{(2n)}$ with $\varepsilon_n \leqslant d^{(n)}(\pi_i^{(n)}, \pi_j^{(n)}) \leqslant s_n$, $\varepsilon_n = 0, 1$.

Summary of Construction II

(i) If $k_1 = 2$ and

$$0 = (a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}) z_1 + (\lambda_2^{(n)} u_n - \lambda^{(n)} b^{(n)}) z_n + \sum_{h=2}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (7)$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, then there exists a $3-(2v, k, \Theta)$ design with

$$\Theta = (a_1 u_{n+1} + \lambda_2^{(n+1)} u_1) z_1 + (\lambda_2^{(n)} u_n) z_n + \sum_{h=2}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

(ii) If $k_1 \geqslant 3$ and

$$0 = (\lambda_2^{(n)} u_n - \lambda^{(n)} b^{(n)}) z_n + \sum_{h=1}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (8)$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, then there exists a $3-(2v, k, \Theta)$ design with

$$\Theta = (\lambda_2^{(n)} u_n) z_n + \sum_{h=1}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

Summary of Construction II (Cont.)

(iii) If $k_1 = 2$ and

$$\begin{aligned} 0 &< (a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}) z_1 + (\lambda_2^{(n)} u_n - \lambda^{(n)} b^{(n)}) z_n \\ &+ \sum_{h=2}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (9) \end{aligned}$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, further if there is a $3-(v, k, \Lambda)$ design having

$$\begin{aligned} \Lambda &= (a_1 u_{n+1} + \lambda_2^{(n+1)} u_1 - \lambda^{(n+1)} b^{(1)}) z_1 + (\lambda_2^{(n)} u_n - \lambda^{(n)} b^{(n)}) z_n \\ &+ \sum_{h=2}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \quad (10) \end{aligned}$$

then there exists a $3-(2v, k, \Theta)$ design with

$$\Theta = (a_1 u_{n+1} + \lambda_2^{(n+1)} u_1) z_1 + (\lambda_2^{(n)} u_n) z_n + \sum_{h=2}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

Summary of Construction II (Cont.)

(iv) If $k_1 \geqslant 3$ and

$$\begin{aligned} 0 &< (\lambda_2^{(n)} u_n - \lambda^{(n)} b^{(n)}) z_n \\ &+ \sum_{h=1}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \end{aligned} \tag{11}$$

with $1 \leqslant z_h \leqslant w_h$ if both D_h and D_{n+h} are simple and $1 \leqslant z_h \leqslant t_h$ if D_h or D_{n+h} is non-simple, further if there is a $3-(v,k,\Lambda)$ design having

$$\begin{aligned} \Lambda &= (\lambda_2^{(n)} u_n - \lambda^{(n)} b^{(n)}) z_n \\ &+ \sum_{h=1}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h) - (\lambda^{(h)} b^{(n+h)} + \lambda^{(n+h)} b^{(h)})\} z_h, \end{aligned} \tag{12}$$

then there exists a $3-(2v,k,\Theta)$ design with

$$\Theta = (\lambda_2^{(n)} u_n) z_n + \sum_{h=1}^{n-1} \{(\lambda_2^{(h)} u_{n+h} + \lambda_2^{(n+h)} u_h)\} z_h.$$

APPLICATIONS

For applications of Constructions I and II we implicitly use the following result and observation.

- **Baranyai-Theorem** The trivial $k-(v,k,1)$ design is (1,1)-resolvable (i.e. having a parallelism) if and only if $k|v$.
- **Block orbits** If $\gcd(v,k)=1$, then the $k-(v,k,1)$ design is $(1,v)$-resolvable. (The resolvable classes are the block orbits of a fixed point free automorphism of order v.)

APPLICATIONS

F1 Construction II with $n = 1$.

v, k : integers with $v > k \geqslant 3$ and $\gcd(v, k) = 1$. D_1: the complete design $3 - (v, k, \binom{v-3}{k-3})$. Then $\lambda^{(1)} = \binom{v-3}{k-3}$, $\lambda_2^{(1)} = \binom{v-2}{k-2}$, $u_1 = k$, $b^{(1)} = v$, and $w_1 = \binom{v-1}{k-1}/k$.

D: $3 - (v, 2k, \Lambda)$.

Construction II yields a simple 3-design $3 - (2v, 2k, \Theta)$ when it holds

$$(\lambda_2^{(1)} u_1 - \lambda^{(1)} b^{(1)}) z_1 = \Lambda,$$

or

$$z_1 = \Lambda / 2\binom{v-3}{k-2} \quad \text{is an integer,}$$

with $z_1 \leqslant \binom{v-1}{k-1}/k$. Then

$$\Theta = \lambda_2^{(1)} u_1 z_1 = \frac{k(v-2)\Lambda}{2(v-k)}.$$

APPLICATIONS

F1 (Cont.)

Take the complete design D: $3-(v,2k,\Lambda) := 3-(v,2k,\binom{v-3}{2k-3})$.

- If
$$z_1 = \binom{v-3}{2k-3}/2\binom{v-3}{k-2} \quad \text{is an integer,}$$
with $z_1 \leqslant \binom{v-1}{k-1}/k$. Then there is a simple $3-(2v,2k,\Theta)$ design with
$$\Theta = \frac{k(v-2)}{2(v-k)}\binom{v-3}{2k-3}.$$

APPLICATIONS

F1 Some special cases: $k = 3, 4, 5$.

1. There exists a simple $3-(2v, 6, \Theta)$ design with
$$\Theta = \frac{3(v-2)}{2(v-3)}\binom{v-3}{3},$$
for $v \equiv 1, 4, 5, 8 \mod 12$.
2. There exists a simple $3-(2v, 8, \Theta)$ design with
$$\Theta = \frac{4(v-2)}{2(v-4)}\binom{v-3}{5},$$
for $v \equiv 1, 5, 7, 11, 15, 17 \mod 20$.
3. There exists a simple $3-(2v, 10, \Theta)$ design with
$$\Theta = \frac{5(v-2)}{2(v-5)}\binom{v-3}{7},$$
for $v \equiv 0, 1, 2, 6 \mod 7$, and $v \equiv 0, 1, 6, 7 \mod 8$, and $\gcd(v, 5) = 1$.

APPLICATIONS

F2 Construction II with $n = 2$, $k_1 = 2$.

v, k : integers with $v > 2k$, $k \geqslant 3$, $\gcd(v, 2k) = 1$ & $\gcd(v, k+1) = 1$.

C_1: $2-(v, 2, 1)$; $\alpha^{(1)} = 0$, $\alpha_2^{(1)} = 1$, $u_1 = 2$, $b^{(1)} = v$, $t_1 = (v-1)/2$,
$a_1 = \frac{1}{k(2k-1)}\binom{v-2}{2k-2}$. D_1 is a union of a_1 copies of C_1.

D_3: $2-(v, 2k, \binom{v-3}{2k-3})$; $\lambda^{(3)} = \binom{v-3}{2k-3}$, $\lambda_2^{(3)} = \binom{v-2}{2k-2}$, $u_3 = 2k$, $b^{(3)} = v$,
$w_3 = \frac{1}{2k}\binom{v-1}{2k-1}$.

D_2: $2-(v, k+1, \binom{v-3}{k-2})$; $\lambda^{(2)} = \binom{v-3}{k-2}$, $\lambda_2^{(2)} = \binom{v-2}{k-1}$, $u_2 = k+1$, $b^{(2)} = v$,
$w_2 = \frac{1}{k+1}\binom{v-1}{k}$.

APPLICATIONS

F2 (Cont.)

Set $A := A_1 z_1 + A_2 z_2$,

where $A_1 = (a_1 u_3 + \lambda_2^{(3)} u_1 - \lambda^{(3)} b^{(1)})$, $A_2 = (\lambda_2^{(2)} u_2 - \lambda^{(2)} b^{(2)})$.
Then

$$\begin{aligned} A_1 &= -\binom{v-3}{2k-3} \frac{v(4k^2-10k+2)+8k}{(2k-1)(2k-2)}, \\ A_2 &= 2\binom{v-3}{k-2} \frac{(v-k-1)}{(k-1)}. \end{aligned}$$

For any integer z_1 with $1 \leqslant z_1 \leqslant w_1$ we have $A = 0$ iff

$$z_2 = -A_1 z_1 / A_2$$

- If z_2 is an integer with $z_2 \leqslant w_2$,then there is a simple $3-(2v, 2(k+1), \Theta)$ design with

$$\Theta = \left(\frac{2k}{k(2k-1)} \binom{v-2}{2k-2} + 2\binom{v-2}{2k-2} \right) z_1 + \binom{v-2}{k-1} z_2.$$

APPLICATIONS

F2 (Cont.)

An example: $z_1 = 1$.
Then

$$z_2 = \binom{v-k-2}{k} \frac{k!}{2.k(k+1)\dots(2k-3)} \frac{v(4k^2-10k+2)+8k}{(2k-1)(2k-2)}.$$

- If z_2 is an integer and $z_2 \leqslant w_2$, then there is a simple $3-(2v, 2(k+1), \Theta)$ design with

$$\Theta = \frac{4k}{(2k-1)}\binom{v-2}{2k-2} + \binom{v-2}{k-1}(k+1).z_2.$$

APPLICATIONS

F2 (Cont.) Two special cases: $k = 3, 4$ with $z_1 = 1$.

1. There exists a simple $3-(2v, 8, \Theta)$ design with

$$\Theta = \frac{7}{30}v(v-2)(v-3)(v-5),$$

   for all $v \equiv 5, 17, 35, 47 \mod 60$.

2. There exists a simple $3-(2v, 10, \Theta)$ design with

$$\Theta = 81v\binom{v-2}{6}/7(v-5),$$

   for all $v \equiv 7, 23, 63, 111, 167, 191, 223, 231, 247 \mod 280$.

APPLICATIONS

F3 Some more examples

1. There exists a simple $3-(2v, 5, \frac{3}{4}(v-2)(v-3))$ design when $v \equiv 2 \mod 6$.
2. There exists a simple $3-(2v, 7, \frac{5}{48}\binom{v-2}{3}(11v-52))$ design for all $v \equiv 4, 76, 112, 148 \mod 180$.
3. There exists a simple $3-(2(2^f+1), 5, 15(2^f-1)$ design for f odd.
4. There exists a simple $3-(2(2^f+1), 6, (2^f-1).m)$ design with $m = 5, 30, 35, 45, 50, 75, 80$ and $\gcd(f, 6) = 1$.

$(1,\sigma)$-RESOLVABILITY

- For each pair (D_i, D_{n+i}) define
$$\sigma^{(i)} = u_i b^{(n+i)} + u_{n+i} b^{(i)}.$$
- For the pair (D_n, D_n) in Construction II define
$$\sigma^{(n)} = u_n b^{(n)}.$$
- Let $m_1, \ldots, m_n$ be integers such that
$$m_i \sigma^{(i)} = m_j \sigma^{(j)} := \sigma \ \text{ for } \ i, j = 1, \ldots, n.$$
- If a $3-(v, 2k, \Lambda)$ design D is required in the construction, it is assumed that D is $(1, \sigma)$-resolvable.

■ Assume that the blocks constructed by using each pair (D_i, D_{n+i}) can be partitioned into $1-(v, 2k, \sigma)$ designs. Then the designs obtained from Constructions I and II are $(1, \sigma)$-resolvable.

$(1, \sigma)$-RESOLVABILITY

Some examples

- The $3-(2v, 6, \frac{1}{4}(v-2)(v-4)(v-5))$ designs in **F1** are $(1, 3v)$-resolvable when $v \equiv 1, 4, 5, 13, 20, 28, 29, 32 \mod 36$.
- The $3-(2v, 8, \Theta)$ designs with $\Theta = \frac{7}{30}v(v-2)(v-3)(v-5)$, and $v \equiv 5, 17, 35, 47 \mod 60$ in **F2** are $(1, 8v)$-resolvable.
- The $3-(2v, 10, \Theta)$ designs with $\Theta = 81v\binom{v-2}{6}/7(v-5)$, and $v \equiv 7, 23, 63, 111, 167, 191, 223, 231, 247$ mod 280 in **F2** are $(1, 10v)$-resolvable, when $16|(v-7)$.
- The $3-(2v, 5, \frac{3}{4}(v-2)(v-4))$ designs in **F3** are $(1, 5v)$-resolvable, when $v \equiv 2, 26, 104, 128$ mod 150.

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