Transitive combinatorial structures invariant under some subgroups of $S(6,2)$

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An incidence structure is an ordered triple $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points, the elements of the set $\mathcal{B}$ are called blocks and $\mathcal{I}$ is called an incidence relation.

- An isomorphism from one incidence structure to other is a bijective mapping of points to points and blocks to blocks which preserves incidence.
- An isomorphism from an incidence structure $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$.
- The set of all automorphisms forms a group called the full automorphism group of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$.

A t-(v, k, λ) design is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

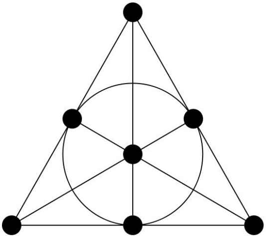

Figure: 2-$(7, 3, 1)$ design

A 2-(v, k, λ) design is called a block design.

Note that this definition allows $\mathcal{B}$ to be a multiset. If $\mathcal{B}$ is a set then $\mathcal{D}$ is called a simple design. If the design $\mathcal{D}$ consists of k copies of some simple design $\mathcal{D}'$ than $\mathcal{D}$ is nonsimple design and it is denoted $\mathcal{D} = k\mathcal{D}'$

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a t-(v, k, λ) design, with $0 \leq s \leq t$. $\mathcal{D}$ is also an s-(v, k, λ_s) design where

$$\lambda_s \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}.$$

Every t-design is also an s-design for $s \leq t$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a finite incidence structure. $\mathcal{G}$ is a graph if each element of $\mathcal{E}$ is incident with exactly two elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices and the elements of $\mathcal{E}$ are called edges.

Two vertices u and v are called adjacent or neighbours if they are incident with the same edge. The number of neighbours of a vertex v is called the degree of v. If all the vertices of the graph $\mathcal{G}$ have the same degree k, then $\mathcal{G}$ is called k-regular.

A graph $\mathcal{G}$ is called a strongly regular graph with parameters (n, k, λ, μ), and denoted by $SRG(n, k, \lambda, \mu)$, if $\mathcal{G}$ is $k-$regular graph with n vertices and if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours.

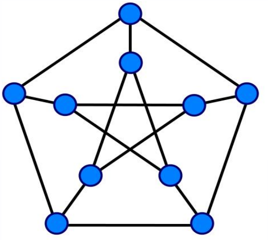

Figure: Petersen Graph

J. D. Key, J. Moori, Codes, Designs and Graphs from the Janko Groups J_1 and J_2, *J. Combin. Math. Combin. Comput.* 40 (2002), 143–159.

- The construction method of primitive symmetric designs and regular graphs for which a stabilizer of a point and a stabilizer of a block are conjugate.

D. Crnković, V. Mikulić, Unitals, projective planes and other combinatorial structures constructed from the unitary groups $U(3, q)$, $q = 3, 4, 5, 7$, *Ars Combin.* 110 (2013), 3–13.

- The construction method of primitive designs and regular graphs for which a stabilizer of a point and a stabilizer of a block are not necessarily conjugate.

- D. Crnković, V. Mikulić, A. Švob, On some transitive combinatorial structures constructed from the unitary group U(3,3), *J. Statist. Plann. Inference* 144 (2014), 19-40.

Theorem

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} G_\alpha.\delta_i$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{g.\Delta_2 : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |G_{\delta_i}.\alpha|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G/\bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design.

Corollary

If a group G acts transitively on the points and the blocks of a 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in the Theorem, i.e., such that Δ_2 is a union of G_α-orbits.

We can use the Theorem to construct 1-design as follows. Let M be a finite group and H_1, H_2, and G be subgroups of M. G acts transitively on the class $ccl_G(H_i),\ i = 1, 2$, by conjugation and

$$|ccl_G(H_1)| = [G : N_G(H_1)] = m,$$

$$|ccl_G(H_2)| = [G : N_G(H_2)] = n.$$

Let us denote the elements of $ccl_G(H_1)$ by $H_1^{g_1}, H_1^{g_2}, \ldots, H_1^{g_m}$, and the elements of $ccl_G(H_2)$ by $H_2^{h_1}, H_2^{h_2}, \ldots, H_2^{h_n}$.

We can construct a 1-design such that:

- the point set of the design is $ccl_G(H_2)$,
- the block set is $ccl_G(H_1)$,
- the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, ..., G_k\} \subset \{H_2^x \cap H_1^y \mid x, y \in G\}$.

Let M be a finite group and H and G be subgroups of M. One can construct regular graph in the following way:

- the vertex set of the graph is $ccl_G(H)$,
- the vertex H^{g_i} is adjacent to the vertex H^{g_j} if and only if $H^{g_i} \cap H^{g_j} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, ..., G_k\} \subset \{H^x \cap H^y \mid x, y \in G\}$.

- We considered transitive structures constructed from a simple group G isomorphic to the symplectic group $S(6,2)$[1]. We described structures constructed on the conjugacy classes of the maximal and second maximal subgroups of the group $S(6,2)$.
- We considered transitive structures constructed from a simple group isomorphic to the unitary group $G \cong U(3,3)$[2]. We described structures constructed on the conjugacy classes of the maximal and second maximal subgroups of the group G.

[1] D. Crnković, V. Mikulić Crnković, A. Švob, On some transitive combinatorial structures and codes constructed from the symplectic group $S(6,2)$, J. Combin. Math. Combin. Comput., to appear.

[2] D. Crnković, V. Mikulić, A. Švob, On some transitive combinatorial structures constructed from the unitary group U(3,3), *J. Statist. Plann. Inference* 144 (2014), 19-40.

The group $S(6,2)$ has 1993 maximal subgroups, and has 8 distinct $S(6,2)-$conjugacy classes of the maximal subgroups $M_1, M_2, ..., M_8$.

Table: Maximal subgroups of the group $S(6,2)$ up to $S(6,2)$-conjugation

Subgroup	Structure of the subgroup	Size	Size of G-conjugacy class
M_8	$U(4,2){:}Z_2$	51840	28
M_7	S_8	40320	36
M_6	$E_{32}{:}S_6$	23040	63
M_5	$U(3,3){:}Z_2$	12096	120
M_4	$E_{64}{:}L(3,2)$	10752	135
M_3	$((E_{16}{:}Z_2) \times E_4){:}(S_3 \times S_3)$	4608	315
M_2	$S_3 \times S_6$	4320	336
M_1	$L(2,8){:}Z_3$	1512	960

We consider structures constructed on the conjugacy classes of the maximal subgroups of the group $S(6, 2)$ under the action of the two not conjugate subgroups, $U(3, 3)$ and $U(4, 2)$.

We do not need to consider conjugacy classes of all maximal subgroups, we can eliminate some of them. We search for all those maximal subgroups of the $S(6, 2)$ which are not conjugate under the action of the groups $U(3, 3)$ and $U(4, 2)$.

Finally, after elimination, we got:

- 14 maximal subgroups of the group $S(6,2)$, which are not conjugate under the action of the subgroup $U(3,3)$,
- 12 maximal subgroups of the group $S(6,2)$ which are not conjugate under the action of the subgroup $U(4,2)$

Table: Maximal subgroups of the group $S(6,2)$ up to $U(3,3)$-conjugation

Group	Structure of the group	Size of the class
N_1^1	$U(4,2){:}Z_2$	28
N_2^1	S_8	36
N_3^1	$E_{32}{:}S_6$	63
N_4^1	$U(3,3){:}Z_2$	1
N_5^1	$U(3,3){:}Z_2$	63
N_6^1	$U(3,3){:}Z_2$	56
N_7^1	$E_{64}{:}L(3,2)$	36
N_8^1	$E_{64}{:}L(3,2)$	36
N_9^1	$E_{64}{:}L(3,2)$	63
N_{10}^1	$((E_{16}{:}Z_2) \times E_4){:}(S_3 \times S_3)$	63
N_{11}^1	$((E_{16}{:}Z_2) \times E_4){:}(S_3 \times S_3)$	252
N_{12}^1	$S_3 \times S_6$	336
N_{13}^1	$L(2,8){:}Z_3$	288
N_{14}^1	$L(2,8){:}Z_3$	672

Table: Maximal subgroups of the group $S(6,2)$ up to $U(4,2)$-conjugation

Group	Structure of the group	Size of the class
N^2_1	$U(4,2){:}Z_2$	27
N^2_2	$U(4,2){:}Z_2$	1
N^2_3	S_8	36
N^2_4	$E_{32}{:}S_6$	36
N^2_5	$E_{32}{:}S_6$	27
N^2_6	$U(3,3){:}Z_2$	120
N^2_7	$E_{64}{:}L(3,2)$	135
N^2_8	$((E_{16}{:}Z_2) \times E_4){:}(S_3 \times S_3)$	270
N^2_9	$((E_{16}{:}Z_2) \times E_4){:}(S_3 \times S_3)$	45
N^2_{10}	$S_3 \times S_6$	216
N^2_{11}	$S_3 \times S_6$	120
N^2_{12}	$L(2,8){:}Z_3$	960

Table: Block designs constructed from the group $S(6,2)$, from the conjugacy classes of maximal subgroups under the action of the subgroup $U(3,3)$

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	$Aut\mathcal{D}$
$\mathcal{D}_1$	$(28, 12, 11)$	yes	$S(6,2)$
$\mathcal{D}_2$	$(28, 4, 1)$	yes	$U(3,3) : Z_2$
$\mathcal{D}_3$	$(28, 4, 4)$	yes	$U(3,3) : Z_2$
$\mathcal{D}_4$	$(28, 10, 40)$	yes	$S(6,2)$
$\mathcal{D}_5$	$(36, 16, 12)$	yes	$S(6,2)$
$\mathcal{D}_6$	$(36, 6, 8)$	yes	$S(6,2)$
$\mathcal{D}_7$	$(63, 31, 15)$	yes	$PGL(6,2)$
$\mathcal{D}_8$	$(36, 15, 6)$	yes	$U(3,3) : Z_2$
$\mathcal{D}_9$	$(63, 31, 15)$	yes	$U(3,3) : Z_2$

Table: Block design constructed from the group $S(6,2)$, from the conjugacy classes of maximal subgroups under the action of the subgroup $U(4,2)$

Block design $\mathcal{D}$	Parameters of $\mathcal{D}$	Simple design	Corresponding simple design	$Aut\mathcal{D}$
$\widetilde{\mathcal{D}_1}$	$(36,8,6)$	yes		$S(6,2)$
$\widetilde{\mathcal{D}_2}$	$(36,15,6)$	yes		$U(4,2):Z_2$
$\widetilde{\mathcal{D}_3}$	$(45,12,9)$	no	$(45,12,3)$	$U(4,2):Z_2$
$\widetilde{\mathcal{D}_4}$	$(45,12,3)$	yes		$U(4,2):Z_2$
$\widetilde{\mathcal{D}_5}$	$(45,12,8)$	yes		$U(4,2):Z_2$

Table: Strongly regular graphs constructed from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(3,3)$

Graph $\mathcal{G}$	Parameters of $\mathcal{G}$	$Aut\mathcal{G}$
$\mathcal{G}_1$	$(63,30,13,15)$	$S(6,2)$
$\mathcal{G}_2$	$(63,30,13,15)$	$U(3,3):Z_2$
$\mathcal{G}_3$	$(36,14,4,6)$	$U(3,3):Z_2$

Table: Strongly regular graphs constructed from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(4,2)$

Graph $\mathcal{G}$	Parameters of $\mathcal{G}$	$Aut\mathcal{G}$
$\widetilde{\mathcal{G}}_1$	$(36,15,6,6)$	$U(4,2):Z_2$
$\widetilde{\mathcal{G}}_2$	$(27,10,1,5)$	$U(4,2):Z_2$
$\widetilde{\mathcal{G}}_3$	$(120,56,28,24)$	$O_8^+(2):Z_2$
$\widetilde{\mathcal{G}}_4$	$(135,64,28,32)$	$O_8^+(2):Z_2$
$\widetilde{\mathcal{G}}_5$	$(45,12,3,3)$	$U(4,2):Z_2$

- We study the linear codes spanned by the incidence matrices of the block designs.
- The linear codes are spanned by incidence vectors of the points and the blocks.
- Additionally, we consider linear codes obtained from the adjacency matrices of the strongly regular graphs.

- The code $C_{\mathbb{F}}$ of the design $\mathcal{D}$ over the finite field $\mathbb{F}$ is the space spanned by the incidence vectors of the blocks over $\mathbb{F}$.
- If $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$; $C_{\mathbb{F}} = \langle v^B \mid B \in \mathcal{B} \rangle$ is a subspace of $\mathbb{F}^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to $\mathbb{F}$.
- Similarly, we can span a code by the incidence vectors of the points over some finite field $\mathbb{F}$.
- All our codes will be linear codes, i.e. subspaces of the ambient vector space; If a code C, over a field of order q, is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to show this information.
- The code of a graph $\mathcal{G}$ over the finite field $\mathbb{F}$ is the row span of an adjacency matrix A over the field $\mathbb{F}$.

If A is an incidence matrix of a 2-(v, k, λ) design $\mathcal{D}$ and p is a prime that does not divide $r - \lambda$, then $\text{rank}_p(A) \geq v - 1$.

If $\text{rank}_p(A) < v - 1$ then p divides $r - \lambda$, hence the code of a design $\mathcal{D}$ is interesting only when p divides $r - \lambda$.

Table: Non-trivial codes, spanned by the blocks of the incidence matrices of the designs (from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(3,3)$)

Design	Parameters	$\lvert\mathrm{Aut}(C)\rvert$	$\mathrm{Aut}(C)$	Self-orthogonal
$\mathcal{D}_1$	$[28,7,12]_2$	1451520	$S(6,2)$	yes
$\mathcal{D}_2$	$[28,21,4]_2$	1451520	$S(6,2)$	no
$\mathcal{D}_5$	$[36,7,16]_2$	1451520	$S(6,2)$	yes
$\mathcal{D}_6$	$[36,21,6]_2$	1451520	$S(6,2)$	no
$\mathcal{D}_7$	$[63,7,31]_2$	20158709760	$PGL(6,2)$	no
$\mathcal{D}_8$	$[36,14,12]_3$	2903040	$O(7,2) \times Z_2$	yes
$\mathcal{D}_9$	$[63,15,6]_2$	12096	$U(3,3) : Z_2$	no

Table: Non-trivial codes, spanned by the points of the incidence matrices of the designs (from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(3,3)$)

Design	Parameters	$\lvert\mathrm{Aut}(C)\rvert$	$\mathrm{Aut}(C)$	Self-orthogonal
$\mathcal{D}_1$	$[63,7,27]_2$	1451520	$S(6,2)$	no
$\mathcal{D}_2$	$[63,21,9]_2$	12096	$U(3,3):Z_2$	no
$\mathcal{D}_3$	$[252,21,36]_2$	12096	$U(3,3):Z_2$	yes
$\mathcal{D}_4$	$[336,21,96]_2$	1451520	$S(6,2)$	yes
$\mathcal{D}_4$	$[336,27]_5$	1451520	$S(6,2)$	yes
$\mathcal{D}_5$	$[63,7,28]_2$	1451520	$S(6,2)$	yes
$\mathcal{D}_6$	$[336,21,56]_2$	1451520	$S(6,2)$	yes
$\mathcal{D}_6$	$[336,35,56]_3$	2903040	$O(7,2)\times Z_2$	no

Table: Non-trivial codes, spanned by the blocks of the incidence matrices of the designs (from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(4,2)$)

Design	Parameters	$\lvert \mathrm{Aut}(C) \rvert$	$\mathrm{Aut}(C)$	Self-orthogonal
$\widetilde{\mathcal{D}_1}$	$[36,15,8]_2$	1451520	$S(6,2)$	no
$\widetilde{\mathcal{D}_5}$	$[45,14,12]_2$	51840	$U(4,2):Z_2$	yes

Table: Non-trivial codes, spanned by the points of the incidence matrices of the designs (from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(4,2)$)

Design	Parameters	$\|\mathrm{Aut}(C)\|$	$\mathrm{Aut}(C)$	Self-orthogonal
$\widetilde{\mathcal{D}_1}$	$[135,15,30]_2$	1451520	$S(6,2)$	yes
$\widetilde{\mathcal{D}_1}$	$[135,36,15]_3$	2903040	$O(7,2)\times Z_2$	yes
$\widetilde{\mathcal{D}_2}$	$[36,15,9]_3$	103680	$(U(4,2):Z_2)\times Z_2$	yes
$\widetilde{\mathcal{D}_4}$	$[45,15,12]_3$	103680	$(U(4,2):Z_2)\times Z_2$	yes
$\widetilde{\mathcal{D}_5}$	$[120,14,32]_2$	51840	$U(4,2):Z_2$	yes
$\widetilde{\mathcal{D}_5}$	$[120,44,18]_3$	103680	$(U(4,2):Z_2)\times Z_2$	no

Table: Non-trivial codes, spanned by the rows of the adjacency matrices of the graphs (from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(3,3)$)

Graph	Parameters	$\|\mathrm{Aut}(C)\|$	$\mathrm{Aut}(C)$	Self-orthogonal
$\mathcal{G}_2$	$[36,8,14]_2$	12096	$U(3,3):Z_2$	yes
$\mathcal{G}_3$	$[63,27,12]_3$	24192	$(U(3,3):Z_2)\times Z_2$	no

Table: Non-trivial codes, spanned by the rows of the adjacency matrices of the graphs (from the group $S(6,2)$ from the conjugacy classes of maximal subgroups under the action of the subgroup $U(4,2)$)

Graph	Parameters	$\|\mathrm{Aut}(C)\|$	$\mathrm{Aut}(C)$	Self-orthogonal
$\widetilde{\mathcal{G}}_3$	$[120,8,56]_2$	348364800	$O_8^+(2):Z_2$	yes

Thank you for your attention!

Transitive combinatorial structures invariant
under some subgroups of $S(6, 2)$

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