Subspace codes in $PG(2n-1, q)$

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(joint work with A. Cossidente)

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Preliminaries

Let V be an n–dimensional vector space over $GF(q)$,

$PG(n-1,q)$

The number of $(r-1)$–dimensional projective subspaces of $PG(n-1,q)$ is

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{(q^n-1)\cdot\ldots\cdot(q^{n-r+1}-1)}{(q^r-1)\cdot\ldots\cdot(q-1)}$$

$$q^{n-1}+q^{n-2}+\cdots+q+1$$

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Codes in Projective Spaces

- $\mathcal{P}_q$ set of all subspaces of $PG(n-1, q)$,
- $\mathcal{G}_q(n, k)$ set of all $(k-1)$–dimensional subspaces of $PG(n-1, q)$, $1 \leq k \leq n$, *Grassmannian*,
- $d_s(U, U') = \dim(U + U') - \dim(U \cap U')$ *subspace distance*.

$(\mathcal{P}_q, d_s)$, $(\mathcal{G}_q(n, k), d_s)$ are metric spaces

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Definition

An $(n, M, d; k)_q$ constant–dimension subspace code (CDC) is a set $\mathcal{C}$ of k–subspaces of V, where $|\mathcal{C}| = M$ and minimum subspace distance
$d_s(\mathcal{C}) = \min\{d_s(U, U') \mid U, U' \in \mathcal{C}, U \neq U'\} = d$.

$\mathcal{A}_q(n, d; k)$ the maximum size of an $(n, M, d; k)_q$ CDC.

Finite Geometry's language

An $(n, M, 2\delta; k)_q$ constant–dimension subspace code, $1 < \delta \leq k$, is a collection $\mathcal{C}$ of $(k-1)$–dimensional projective subspaces of $PG(n-1, q)$ such that every $(k-\delta)$–dimensional projective subspace of $PG(n-1, q)$ is contained in at most a member of $\mathcal{C}$ and $|\mathcal{C}| = M$.

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Known results

$\delta = k$, i.e., $(n, M, 2k; k)_q$ CDC

Largest partial $(k-1)$–spread in $PG(n-1, q)$,

A. Beutelspacher, Partial spreads in finite projective spaces and partial designs, *Math. Z.* 145 (1975), 211-229.

$\delta = 2$, i.e., $(n, M, 4; 2)_q$ CDC

Maximum number of planes in $PG(n-1, q)$ mutually intersecting in at most one point

T. Honold, M. Kiermaier, S. Kurz, Optimal binary subspace codes of length 6, constant dimension 3 and minimum distance 4, *Contemp. Math.* 632 (2015), 157-176.

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MRD codes

$(\mathcal{M}_{n\times n}(q), d_r)$ is a metric space

$\mathcal{M}_{n\times n}(q)$ $n \times n$ matrices over the finite field $GF(q)$,

$d_r(A, B) = rk(A - B)$, *rank distance*

$$q^{n(n-d+1)}$$

A code $\mathcal{C} \subset \mathcal{M}_{m\times n}(q)$, $|\mathcal{C}| = q^{n(n-d+1)}$ is said to be a q–ary (n, n, k) *maximum rank distance code* (*MRD*), where $k = n - d + 1$.

A non–empty subset of $\mathcal{M}_{n\times n}(q)$ such that all elements have rank r and minimum rank distance d is called an (n, n, d, r) *constant–rank code* (CRC) of constant rank r.

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$$\mathcal{A} \; q^{n(n-1)/2} \text{ skew–symmetric matrices of } \mathcal{M}_{n\times n}(q)$$

Remark

There exists an $(n, n, n-1)$ MRD code, say $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$.

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Lifting MRD codes

$A \in \mathcal{M}_{n\times n}(q)$, $L(A) = \langle$rows of $(I_n|A) \in \mathcal{M}_{n\times 2n}(q)\rangle$,

$L(A)$ is an $(n-1)$–dim. of $PG(2n-1, q)$, *lifted of A*.

$$\mathcal{L}_1 = \{L(A) | A \in \mathcal{M}\}$$

$\forall A, B \in \mathcal{M}$, $\dim(L(A) + L(B)) = rk \begin{pmatrix} I_n & A \\ I_n & B \end{pmatrix} =$

$rk \begin{pmatrix} I_n & A \\ 0_n & A-B \end{pmatrix} = n + rk(A-B) \geq n+2.$

$$\dim(L(A) \cap L(B)) \leq n-2$$

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$\forall A, B \in \mathcal{M}$, $\dim(L(A)+L(B)) = rk \begin{pmatrix} I_n & A \\ I_n & B \end{pmatrix} =$
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U_i point with 1 in the i–th position, 0 elsewhere,

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$S = \langle U_{n+1}, \ldots, U_{2n} \rangle$

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$$\begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{j=2}^{r} (-1)^{(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{\binom{r-j}{2}} (q^{n(j-1)} - 1).$$

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Subspace Codes in $PG(2n-1, q)$

Proposition

The set $\bigcup_{i=1}^{n-2} \mathcal{L}_i$ is a $(2n, M, 4; n)_q$ constant–dimension subspace code, where

$$M = q^{n^2-n} + \sum_{r=2}^{n-2} \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{j=2}^{r} (-1)^{(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{\binom{r-j}{2}} (q^{n(j-1)} - 1).$$

$n = 4$

each element in $\mathcal{L}_1$ is disjoint from S,

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The triality quadric $\mathcal{Q}^+(7,q)$

$\mathcal{Q}$ non–degenerate hyperbolic quadric of $PG(7,q)$,

$$\mathcal{Q} : X_1X_5 + X_2X_6 + X_3X_7 + X_4X_8 = 0$$

$$|\mathcal{Q}| = (q+1)(q^2+1)(q^3+1)$$

generators of $\mathcal{Q}$ are solids (3–dim. pr. spaces),

$$2(q+1)(q^2+1)(q^3+1).$$

two *systems of generators* $\longrightarrow \mathcal{M}_1, \mathcal{M}_2$

$$A \in \mathcal{M}_i, A' \in \mathcal{M}_j$$

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$D(X), I(X) \longrightarrow$ set of generators in $\mathcal{M}_1$ disjoint from or meeting X, resp.

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$D(X), I(X) \longrightarrow$ set of generators in $\mathcal{M}_1$ disjoint from or meeting X, resp.

$$\mathcal{M}_1 = D(S) \cup (D(S') \cap I(S)) \cup (I(S) \cap I(S'))$$

$|D(S)| = q^6$,

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If $g \in I(S) \cap I(S')$, then g meets both S, S' in a line,

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S, S' are generators of $\mathcal{Q}_i$, $\mathcal{M}_1^i$, $l_i(S) \cap l_i(S')$

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base locus of $\mathcal{F}$ are the points covered by elements of $\mathcal{G}$.

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$$\mathcal{D}_S = \{A \cup S \mid A \in \mathcal{G}\}, \mathcal{D}_{S'} = \{A \cup S' \mid A \in \mathcal{G}\}$$

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Subspace Codes in $PG(7, q)$

r line of S, $\exists! r'$ line of S' s.t. $\langle r, r' \rangle \in \mathcal{M}_1$

if $r \notin \mathcal{D}_S$, then $r' \notin \mathcal{D}_{S'}$

$l_1, \dots, l_{q+1} \in \mathcal{D}_S$ incident with r,

$\mathcal{D}_r$ Desarguesian line–spread of S,

$\mathcal{D}_r \cap \mathcal{D}_S = \{l_1, \dots, l_{q+1}\}$

$\mathcal{X}$ solids of type $\langle r', t \rangle$, $t \in \mathcal{D}_r \setminus \{l_1, \dots, l_{q+1}\}$

$|\mathcal{X}| = (q^2 - q)(q^2 + 1)(q^2 + q) = q^6 - q^2$

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$\mathcal{X}$ solids of type $\langle r', t \rangle$, $t \in \mathcal{D}_r \setminus \{l_1, \ldots, l_{q+1}\}$

$|\mathcal{X}| = (q^2 - q)(q^2 + 1)(q^2 + q) = q^6 - q^2$

r line of S, $\exists! r'$ line of S' s.t. $\langle r, r' \rangle \in \mathcal{M}_1$

if $r \notin \mathcal{D}_S$, then $r' \notin \mathcal{D}_{S'}$

$l_1, \ldots, l_{q+1} \in \mathcal{D}_S$ incident with r,

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$$q^{12} + q^2(q^2+1)^2(q^2+q+1) + 1$$

solids of $PG(7, q)$ mutually intersecting in at most a line.

$$\mathcal{A}_q(8, 4; 4) \geq q^{12} + q^2(q^2+1)^2(q^2+q+1) + 1$$

T. Etzion, N. Silberstein, Codes and Designs Related to Lifted MRD Codes, *IEEE Trans. Inform. Theory* 59 (2013), no. 2, 1004-1017.

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THANKS