On q-analogs of 3-(v, k, λ_3) designs

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Theorem (Krčadinac, Nakić, Pavčević, 2014)

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *be a* t-(v, k, λ_t) *design with tactical decomposition*

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Then the coefficients of $\mathcal{R} = [\rho_{ij}]$ *and* $\mathcal{K} = [\kappa_{ij}]$ *satisfy*

$$\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_1 j}^{m_1 - 1} \kappa_{i_2 j}^{m_2} \cdots \kappa_{i_s j}^{m_s} =$$

$$\sum_{\omega_1=1}^{m_1} \sum_{\omega_2=1}^{m_2} \cdots \sum_{\omega_s=1}^{m_s} \lambda_{\omega_1+\cdots+\omega_s} \left\{ {m_1 \atop \omega_1} \right\} (|\mathcal{P}_{i_1}| - 1)_{\omega_1 - 1} \prod_{j=2}^{s} \left\{ {m_j \atop \omega_j} \right\} (|\mathcal{P}_{i_j}|)_{\omega_j}.$$

Question: can this theorem be q-analogized?

Definition

A t-(v, k, λ_t) *design* is a finite incidence structure $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$ is a set of v elements called *points*,
- $\mathcal{B}$ is a set of k-subsets of $\mathcal{P}$ called *blocks*,
- every set of t points is contained in precisely λ_t blocks.

- t-(v, k, λ_t) design $\Rightarrow$ s-(v, k, λ_s) design with $s < t$ and $\lambda_s = \lambda_t \binom{v-s}{t-s} / \binom{k-s}{t-s}$.

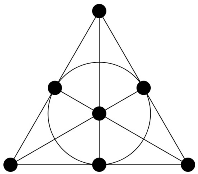


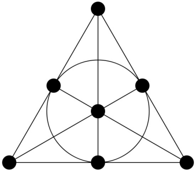
Figure: The Fano plane. 2-$(7, 3, 1)$ design.

Definition

A t-$(v, k, \lambda_t)_q$ *design* is finite set $\mathcal{B}$, where

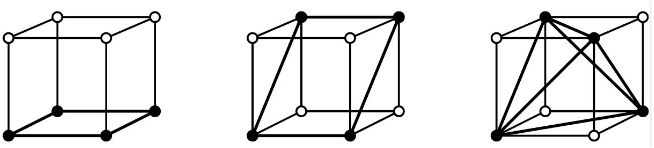
- $\mathcal{B}$ is a set of k-subspaces of vector space $\mathbb{F}_q^v$ called *blocks*,
- every t-subspace of $\mathbb{F}_q^v$ is contained in precisely λ_t blocks.

- t-$(v, k, \lambda_t)_q$ design $\Rightarrow$ s-$(v, k, \lambda_s)_q$ design with $s < t$ and $\lambda_s = \lambda_t \begin{bmatrix} v-s \\ t-s \end{bmatrix}_q / \begin{bmatrix} k-s \\ t-s \end{bmatrix}_q$.



Question: does a q-analog of the Fano plane exist? (Braun, Kiermaier, Nakić, 2015)

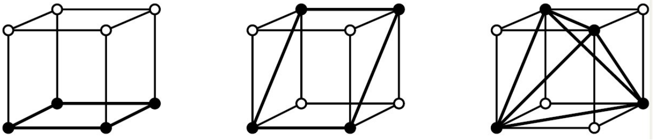
Example: 3-$(8, 4, 1)$ design



- Incidence matrix

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
p_1	1	0	1	0	0	1	1	0	0	0	1	1	0	1
p_2	0	1	1	0	1	0	0	0	1	0	1	0	1	1
p_3	1	0	0	1	1	0	0	1	0	0	1	1	1	0
p_4	0	1	0	1	0	1	0	0	0	1	0	1	1	1
p_5	1	0	1	0	0	1	0	1	1	1	0	0	1	0
p_6	0	1	1	0	1	0	1	1	0	1	0	1	0	0
p_7	1	0	0	1	1	0	1	0	1	1	0	0	0	1
p_8	0	1	0	1	0	1	1	1	1	0	1	0	0	0

Example: 3-$(8,4,1)$ design



- Incidence matrix

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
p_1	1	0	1	0	0	1	1	0	0	0	1	1	0	1
p_2	0	1	1	0	1	0	0	0	1	0	1	0	1	1
p_3	1	0	0	1	1	0	0	1	0	0	1	1	1	0
p_4	0	1	0	1	0	1	0	0	0	1	0	1	1	1
p_5	1	0	1	0	0	1	0	1	1	1	0	0	1	0
p_6	0	1	1	0	1	0	1	1	0	1	0	1	0	0
p_7	1	0	0	1	1	0	1	0	1	1	0	0	0	1
p_8	0	1	0	1	0	1	1	1	1	0	1	0	0	0

$$[\rho_{ij}] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix} \qquad [\kappa_{ij}] = \begin{bmatrix} 2 & 2 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{bmatrix}$$

Definition

A *tactical decomposition* of a design $(\mathcal{P}, \mathcal{B})$ is any partition

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$$

with the property that there exist nonnegative integers ρ_{ij} and κ_{ij} such that

- each point of $\mathcal{P}_i$ lies in precisely ρ_{ij} blocks of $\mathcal{B}_j$,
- and each block of $\mathcal{B}_j$ contains precisely κ_{ij} points from $\mathcal{P}_i$.

Matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ are called *tactical decomposition matrices*.

- Orbits of $\mathcal{P}$ and orbits of $\mathcal{B}$ under an action of $G \leq \mathrm{Aut}\, \mathcal{D}$ form a tactical decomposition of $\mathcal{D}$.

- $(\mathcal{P}, \mathcal{B})$ is a 2-(v, k, λ_2) design with tactical decomposition

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

1. $\sum_{j=1}^{n} \rho_{ij} = \lambda_1, \quad \sum_{i=1}^{m} \kappa_{ij} = k$

2. $\sum_{j=1}^{n} \rho_{lj} \kappa_{rj} = \begin{cases} \lambda_1 + (|\mathcal{P}_r| - 1) \cdot \lambda_2, & l = r, \\ |\mathcal{P}_r| \cdot \lambda_2, & l \neq r. \end{cases}$

$[\rho_{ij}]$	$\mathcal{B}_1$	$\cdots$	$\mathcal{B}_n$
$\vdots$			
$\mathcal{P}_l$	ρ_{l1}	$\cdots$	ρ_{ln}
$\vdots$			
$\vdots$			

$[\kappa_{ij}]$	$\mathcal{B}_1$	$\cdots$	$\mathcal{B}_n$
$\vdots$			
$\vdots$			
$\mathcal{P}_r$	κ_{r1}	$\cdots$	κ_{rn}
$\vdots$			

Theorem (Krčadinac, Nakić, Pavčević, 2014)

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *be a* t-(v, k, λ_t) *design with tactical decomposition*

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

Then the coefficients of $\mathcal{R} = [\rho_{ij}]$ *and* $\mathcal{K} = [\kappa_{ij}]$ *satisfy*

$$\sum_{j=1}^{n} \rho_{i_1 j} \kappa_{i_1 j}^{m_1 - 1} \kappa_{i_2 j}^{m_2} \cdots \kappa_{i_s j}^{m_s} = \sum_{\omega_1=1}^{m_1} \sum_{\omega_2=1}^{m_2} \cdots \sum_{\omega_s=1}^{m_s} \lambda_{\omega_1 + \cdots + \omega_s} \left\{ {m_1 \atop \omega_1} \right\} (|\mathcal{P}_{i_1}| - 1)_{\omega_1 - 1} \prod_{j=2}^{s} \left\{ {m_j \atop \omega_j} \right\} (|\mathcal{P}_{i_j}|)_{\omega_j}.$$

Open problem: the existence of a 3-$(16, 7, 5)$ design

Theorem (Nakić, 2015)

If a 3-$(16, 7, 5)$ design exists, then it is either rigid or its full automorphism group is a 2-group.

- Ψ - the set of 1-spaces of $\mathbb{F}_q^v$

Definition

A *tactical decomposition* of a design $\mathcal{B}$ over finite field with parameters t-$(v, k, \lambda_t)_q$ is any partition

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n$$

with the property that there exist nonnegative integers ρ_{ij} and κ_{ij} such that

- each point of Ψ_i lies in precisely ρ_{ij} blocks of $\mathcal{B}_j$,
- and each block of $\mathcal{B}_j$ contains precisely κ_{ij} points from Ψ_i.

Matrices $\mathcal{R} = [\rho_{ij}]$ and $\mathcal{K} = [\kappa_{ij}]$ are called *tactical decomposition matrices*.

- Orbits of Ψ and orbits of $\mathcal{B}$ under an action of $G \leq$ Aut $\mathcal{B}$ form a tactical decomposition of $\mathcal{B}$.

- (Nakić, Pavčević, 2014)
- $\mathcal{B}$ is a $2\text{-}(v, k, \lambda_2)_q$ design with tactical decomposition

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

1. $$\sum_{j=1}^{n} \rho_{ij} = \lambda_1, \quad \sum_{i=1}^{m} \kappa_{ij} = \begin{bmatrix} k \\ 1 \end{bmatrix}_q$$

2. $$\sum_{j=1}^{n} \rho_{lj}\kappa_{rj} = \begin{cases} \lambda_1 + (|\Psi_r| - 1) \cdot \lambda_2, & l = r, \\ |\Psi_r| \cdot \lambda_2, & l \neq r. \end{cases}$$

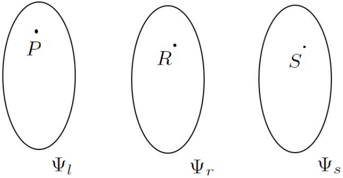
$[\rho_{ij}]$	$\mathcal{B}_1$	$\cdots$	$\mathcal{B}_n$
Ψ_1			
$\vdots$			
Ψ_l	ρ_{l1}	$\cdots$	ρ_{ln}
$\vdots$			
$\vdots$			
Ψ_m			

$[\kappa_{ij}]$	$\mathcal{B}_1$	$\cdots$	$\mathcal{B}_n$
Ψ_1			
$\vdots$			
$\vdots$			
Ψ_r	κ_{r1}	$\cdots$	κ_{rn}
$\vdots$			
Ψ_m			

- $\mathcal{B}$ is a 3-$(v, k, \lambda_3)_q$ design with tactical decomposition

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

$$\sum_{j=1}^{n} \rho_{lj} \kappa_{rj} \kappa_{sj} = \Lambda^1_{lrs} \cdot \lambda_1 + \Lambda^2_{lrs} \cdot \lambda_2 + \Lambda^3_{lrs} \cdot \lambda_3.$$



$\Lambda^i_{lrs} = \#\{(P, R, S) \;:\;$ fixed $P \in \Psi_l, (R, S) \in \Psi_r \times \Psi_s, \dim\langle P, R, S\rangle = i\}, \qquad i = 1, 2, 3.$

- $(\mathcal{P}, \mathcal{B})$ is a 3-(v, k, λ_3) design with tactical decomposition

$$\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_m, \;\; \mathcal{B} = \mathcal{B}_1 \sqcup \cdots \sqcup \mathcal{B}_n.$$

$$\sum_{j=1}^{n} \rho_{lj} \kappa_{rj} \kappa_{sj} = \Lambda^1_{lrs} \cdot \lambda_1 + \Lambda^2_{lrs} \cdot \lambda_2 + \Lambda^3_{lrs} \cdot \lambda_3.$$

Theorem (Krčadinac, Nakić, Pavčević, 2014)

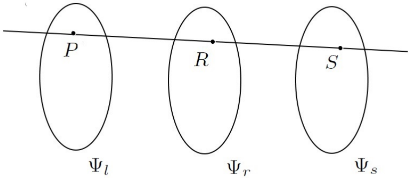
$$\sum_{j=1}^{n} \rho_{lj} \kappa_{rj} \kappa_{sj} =$$

$$= \begin{cases} \lambda_1 + 3\,(|\mathcal{P}_l| - 1) \cdot \lambda_2 + (|\mathcal{P}_l| - 1) \cdot (|\mathcal{P}_l| - 2) \cdot \lambda_3, & \text{for } l = r = s, \\ |\mathcal{P}_r| \cdot |\mathcal{P}_s| \cdot \lambda_3, & \text{for } l \neq r \neq s \neq l, \\ |\mathcal{P}_s| \cdot \lambda_2 + (|\mathcal{P}_r| - 1) \cdot |\mathcal{P}_s| \cdot \lambda_3, & \text{otherwise.} \end{cases}$$

Theorem (De Boeck, Nakić)

$$\sum_{j=1}^{n} \rho_{lj}\kappa_{rj}\kappa_{sj} =$$

$$= \begin{cases} \lambda_1 + \quad \Lambda^2_{lrs} \cdot \lambda_2 + \quad (|\Psi_r| \cdot |\Psi_s| - \Lambda^2_{lrs} - 1) \cdot \lambda_3, & \text{for } l = r = s, \\ \qquad \Lambda^2_{lrs} \cdot \lambda_2 + \quad (|\Psi_r| \cdot |\Psi_s| - \Lambda^2_{lrs}) \cdot \lambda_3, & \text{otherwise.} \end{cases}$$



$$\Lambda^2_{lrs} = \#\{(P, R, S) \ : \ \text{fixed } P \in \mathcal{P}_l, (R, S) \in \mathcal{P}_r \times \mathcal{P}_s, \#\{P, R, S\} = 2\}$$

Lemma

1. $\Lambda^2_{lrs} = \Lambda^2_{lsr}$

2. $|\Psi_l| \cdot \Lambda^2_{lrs} = |\Psi_r| \cdot \Lambda^2_{rls}$

3. $\sum_{s=1}^{m} \Lambda^2_{lrs} = \begin{cases} |\Psi_r| \cdot (q+1) + \frac{q^v - q^2}{q-1} - 1, & l = r, \\ |\Psi_r| \cdot (q+1), & l \neq r. \end{cases}$

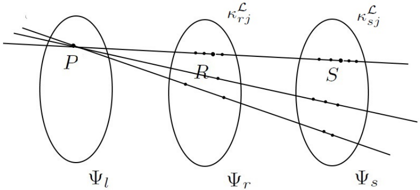
Lemma

The set of 2-subspaces of $\mathbb{F}_q^v$ is a $2-(v,2,1)_q$ design $\mathcal{L}$. Group $G \leq P\Gamma L(\mathbb{F}_q^v)$ acts on $\mathcal{L}$ inducing tactical decomposition

$$\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_m, \qquad \mathcal{L} = \mathcal{L}_1 \sqcup \cdots \sqcup \mathcal{L}_\omega$$

with tactical decomposition matrices $[\rho^{\mathcal{L}}_{ij}]$ and $[\kappa^{\mathcal{L}}_{ij}]$. Then

$$\Lambda^2_{lrs} = \begin{cases} \sum_{j=1}^{\omega} \rho^{\mathcal{L}}_{lj}\kappa^{\mathcal{L}}_{rj}\kappa^{\mathcal{L}}_{sj} - \lambda_1, & \text{for } l = r = s, \\ \sum_{j=1}^{\omega} \rho^{\mathcal{L}}_{lj}\kappa^{\mathcal{L}}_{rj}\kappa^{\mathcal{L}}_{sj}, & \text{otherwise.} \end{cases}$$



Thank you for your attention!