ON THE EXISTENCE SPREADS IN PROJECTIVE HJELMSLEV GEOMETRIES

Ivan Landjev

New Bulgarian University

– ALCOMA 2015, Kloster Banz, 15.-20.03.2015 –

1. Modules over finite chain rings

Definition. A ring (associative, $1 \neq 0$, ring homomorphisms preserving 1) is called a **left (right) chain ring** if the lattice of its left (right) ideals forms a chain.

$$R > \operatorname{rad} R > (\operatorname{rad} R)^2 > \ldots > (\operatorname{rad} R)^{m-1} > (\operatorname{rad} R)^m = (0).$$

- m – the **length** of R;
- $\mathbb{F}_q$ – the **residue field** of R, $q = p^s$;
- p^r – the **characteristic** of R.

Theorem. Let R be a finite chain ring of nilpotency index m. For any finite module ${}_R M$ there exists a uniquely determined partition

$$\lambda = (\lambda_1, \ldots, \lambda_k) \vdash \log_q |M|,$$

$0 \leq \lambda_i \leq m$, such that

$${}_R M \cong R/(\mathrm{rad}\, R)^{\lambda_1} \oplus \ldots \oplus R/(\mathrm{rad}\, R)^{\lambda_k}.$$

The partition λ is called the **shape** of ${}_R M$.

The number k is called the **rank** of ${}_R M$.

2. Projective Hjelmslev Geometries

- $M = {}_RR^n$; $M^* := M \setminus \theta M$; $\theta \in \operatorname{rad} R \setminus (\operatorname{rad} R)^2$
- $\mathcal{P} = \{Rx \mid x \in M^*\}$;
- $\mathcal{L} = \{Rx + Ry \mid x, y$ linearly independent$\}$;
- $I \subseteq \mathcal{P} \times \mathcal{L}$ – incidence relation;
- $\bigcirc_i$ - **neighbour relation**:

$$X \bigcirc_i Y \text{ iff } X = Y + \theta^i R^n,$$

for $i = 1, \ldots, m$.

Definition. The incidence structure $\Pi = (\mathcal{P}, \mathcal{L}, I)$ with neighbour relations $\bigcirc_i$ is called the (**left**) **projective Hjelmslev geometry** over the chain ring R.

Definition. A set of points H in the projective Hjelmslev space Π is called a **Hjelmlsev subspace** if for any two points $x, y \in H$ there is at least one line incident with both of them which is entirely contained in H.

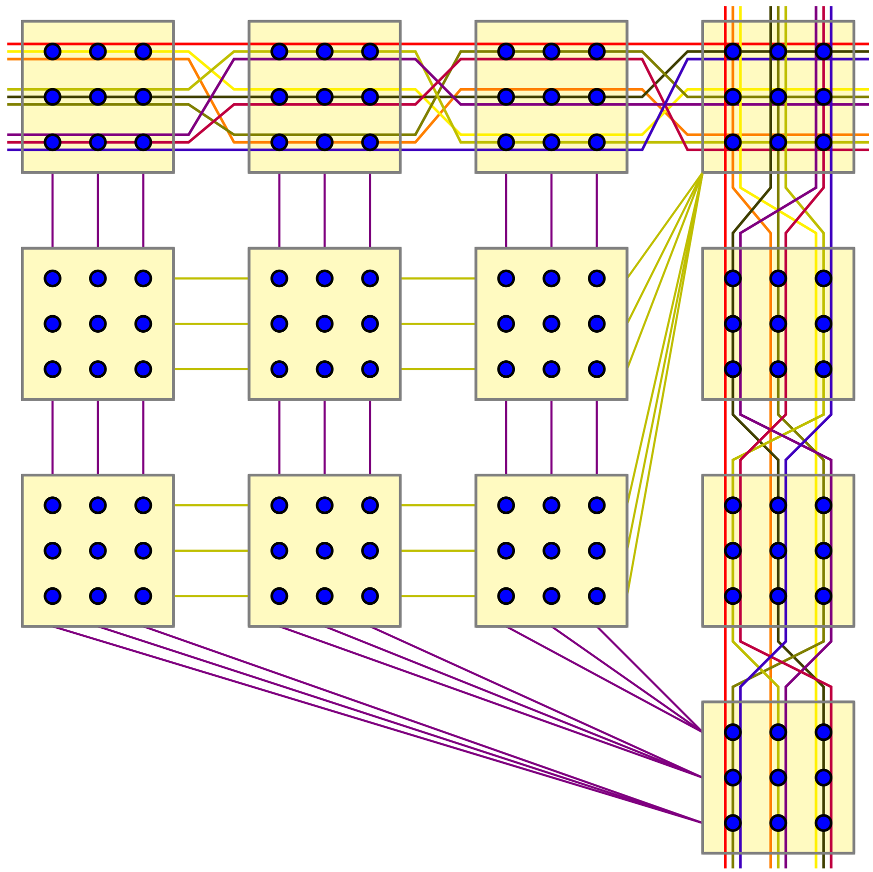
Definition. A set of points H in the projective Hjelmlsev space Π is called a **subspace** if it is the intersection of Hjelmlsev subspaces.

Hjelmslev subspaces $\longrightarrow$ free submodules of ${}_RR^n$

subspaces $\longrightarrow$ submodules of ${}_RR^n$ with at least one free submodule

subspace of type λ $\longrightarrow$ submodule of type λ

$\mathrm{PHG}(\mathbb{Z}_9^3)$

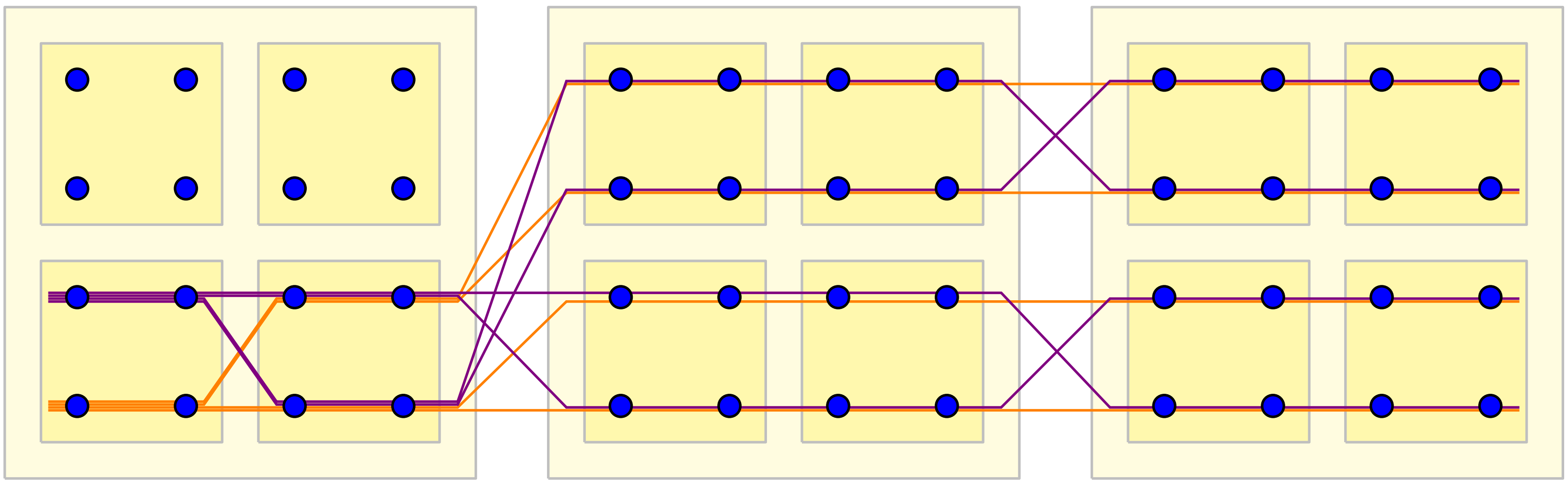


- $\mathcal{S}_0$ – a Hjelmslev subspace with $\dim \mathcal{S}_0 = k-1$ in $\mathrm{PHG}({}_R R^n)$;
- $\mathfrak{P} = \{\mathcal{S} \cap [X]^{(m-i)} \mid X \frown_i \mathcal{S}_0, \mathcal{S} \in [\mathcal{S}_0]^{(i)}\}$;
- $\mathfrak{L}$ – the set of all lines incident with at least one "point" from $[\mathcal{S}_0]$;
- $\mathfrak{I} \subseteq \mathfrak{P} \times \mathfrak{L}$.

Theorem. The incidence structure $(\mathfrak{P}, \mathfrak{L}, \mathfrak{I})$ can be imbedded isomorphically into $\mathrm{PHG}({}_{R/\operatorname{rad}^{m-i} R}(R/\operatorname{rad}^{m-i} R)^n)$. The missing part contains the points of an $(n-k-1)$-dimensional Hjelmslev space over $R/\operatorname{rad}^{m-i} R$.

In particular, for $i = m-1$, $(\mathfrak{P}, \mathfrak{L}, \mathfrak{I})$ can be imbedded isomorphically into $\mathrm{PG}(n-1, q)$.

A Neighbour class of lines in $\mathrm{PHG}(\mathbb{Z}_8^3)$



Theorem. Let ${}_RM$ be a module of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. For every sequence $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \geq \ldots \geq \mu_n \geq 0$, satisfying $\mu \leq \lambda$ the module ${}_RM$ has exactly

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q^m} = \prod_{i=1}^{m} q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix}\lambda'_i - \mu'_{i+1}\\ \mu'_i - \mu'_{i+1}\end{bmatrix}_q$$

submodules of shape μ. In particular, the number of free rank s submodules of ${}_RM$ equals

$$q^{s(\lambda'_1 - s) + \ldots + s(\lambda'_{m-1} - s)} \cdot \begin{bmatrix}\lambda'_m\\ s\end{bmatrix}_q.$$

Here

$$\begin{bmatrix}n\\ k\end{bmatrix}_q = \frac{(q^n - 1)\ldots(q^{n-k+1} - 1)}{(q^k - 1)\ldots(q - 1)}.$$

are the Gaussian coefficients.

3. Spreads

Definition. A k-**spread** of the projective Hjelmslev geometry $\mathrm{PHG}({}_R R^{n+1})$ is a set $\mathcal{S}$ of k-dimensional Hjelmslev subspaces such that every point is contained in exactly one subspace of $\mathcal{S}$.

Theorem. Let R be a chain ring with $|R| = q^m$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. There exists a spread $\mathcal{S}$ of k-dimensional Hjelmslev subspaces of $\mathrm{PHG}({}_R R^{n+1})$ if and only if $k+1$ divides $n+1$.

Let $\kappa = (\kappa_1, \ldots, \kappa_n)$ with $m = \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n \geq 0$.

Definition. A **κ-spread** of the projective Hjelmslev geometry $\mathrm{PHG}(R_R^n)$ is a set $\mathcal{S}$ of subspaces of type κ such that every point is contained in exactly one subspace of $\mathcal{S}$.

Theorem. Let there exist a κ-spread in $\mathrm{PHG}({}_RR^n)$, where $R/\operatorname{rad} R \cong \mathbb{F}_q$, $|R| = q^m$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ with

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = a.$$

Then there exists a μ-spread with $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i = \lambda_i - a$, in the geometry $\mathrm{PHG}({}_SS^n)$, where $S \cong R/\operatorname{rad}^{m-a} R$.

Let $\kappa = (\kappa_1, \ldots, \kappa_n)$, where $m = \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n = 0$.

Theorem. Let R be a finite chain ring and let $\Pi = \mathrm{PHG}({}_R R^n)$. If there exists a κ-spread in Π then $\begin{bmatrix} \kappa \\ 1 \end{bmatrix}_{q^m}$ divides $\begin{bmatrix} n \\ 1 \end{bmatrix}_{q^m}$.

Note: this necessary condition is not always sufficient.

Take a chain ring R with length 2 and residue field $\mathbb{F}_q$.

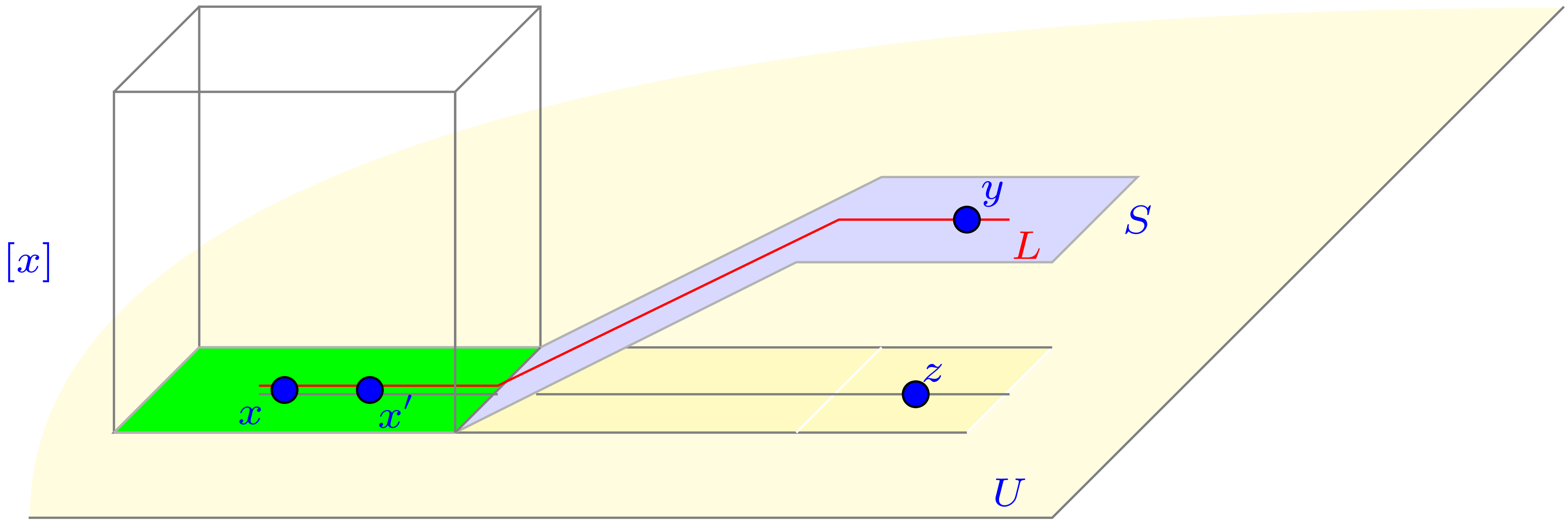
Take $\kappa = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-1},0)$.

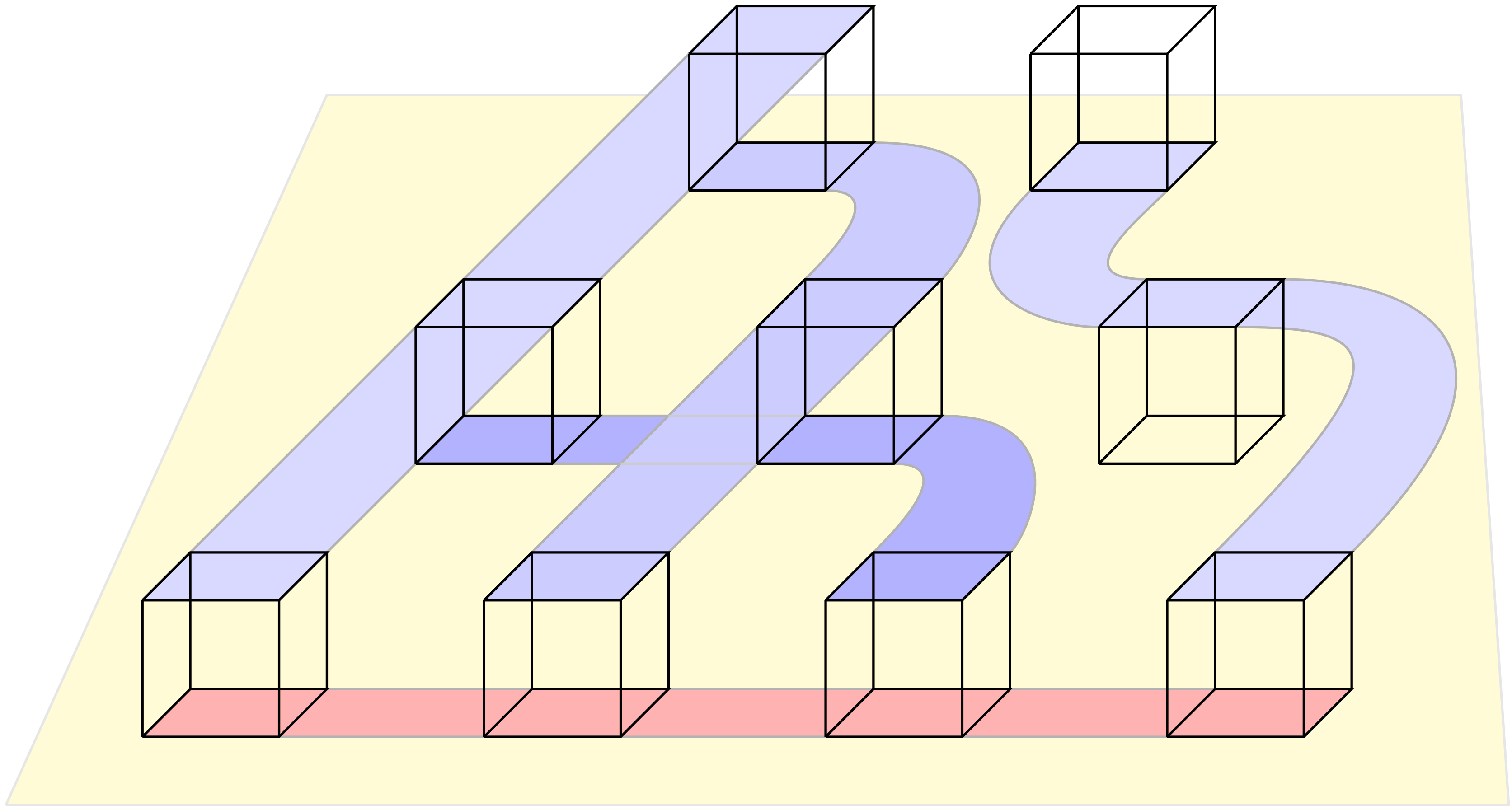
The number of points in a subspace of shape κ is $q^{n-2}\frac{q^{\frac{n}{2}}-1}{q-1}$ and divides the number of points in $\mathrm{PHG}({}_R R^n)$ which is $q^{n-1}\frac{q^n-1}{q-1}$.

Theorem. Let R be a chain ring of nilpotency index 2. Let $\Pi = \mathrm{PHG}({}_R R^n)$. There exists no κ-spread of Π for $\kappa = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-1},0)$.

Corollary. There exists no κ-spread of $\mathrm{PHG}({}_R R^4)$ with $\kappa = (2,2,1,0)$.

- $\Pi = \mathrm{PHG}({}_R R^4)$, $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$
- S – a subspace of shape $(2, 2, 1, 0)$
- U – an Hjelmslev subspace of shape $(2, 2, 2, 0)$
- Observation: if $[x] \cap S \subseteq [x] \cap U$ for some point $x \in \mathcal{P}$ then $S \subseteq [U]$.


$[x]$
y
S
L
z
x
x'
U



More generally:

Theorem. Let R be a chain ring of length m. Let $\Pi = \mathrm{PHG}({}_RR^n)$. There exists no κ-spread of Π for $\kappa = (\underbrace{m,\ldots,m}_{n/2},\underbrace{m-1,\ldots,m-1}_{n/2-1},0)$.

Theorem. Let R be a finite chain ring of length 2 and let $\Pi = \mathrm{PHG}({}_RR^n)$ be the corresponding (left) projective Hjelmslev space. There exists no κ-spread of $\Pi = \mathrm{PHG}({}_RR^n)$ with $\kappa = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-a},\underbrace{0,\ldots,0}_{a})$, where $1 \le a \le \frac{n}{2}-1$.

For chain rings of length 2 and dimension of the free part equal to $n/2$, we have:

Shape	Existence
$(\underbrace{2,\ldots,2}_{n/2},\underbrace{0,\ldots,0}_{n/2})$	YES
$(\underbrace{2,\ldots,2}_{n/2},1,\underbrace{0,\ldots,0}_{n/2-1})$	NO
$(\underbrace{2,\ldots,2}_{n/2},1,1\,\underbrace{0,\ldots,0}_{n/2-2})$	NO
$\ldots$	$\ldots$
$(\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-1},0)$	NO
$(\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2})$	YES

4. Further non-existence results

Does there exist a κ-spread for $\tau = (3,3,1,0)$, in $\mathrm{PHG}({}_R R^4)$, where R is a chain ring with $|R| = q^3$,$R/\operatorname{rad} R \cong \mathbb{F}_q$?

# of points in a subspace of shape $(3,3,1,0)$= $q^3(q+1)$

# of subspaces in the spread $= \dfrac{q^6(q^3+q^2+q+1)}{q^3(q+1)} = q^5 + q^3$.

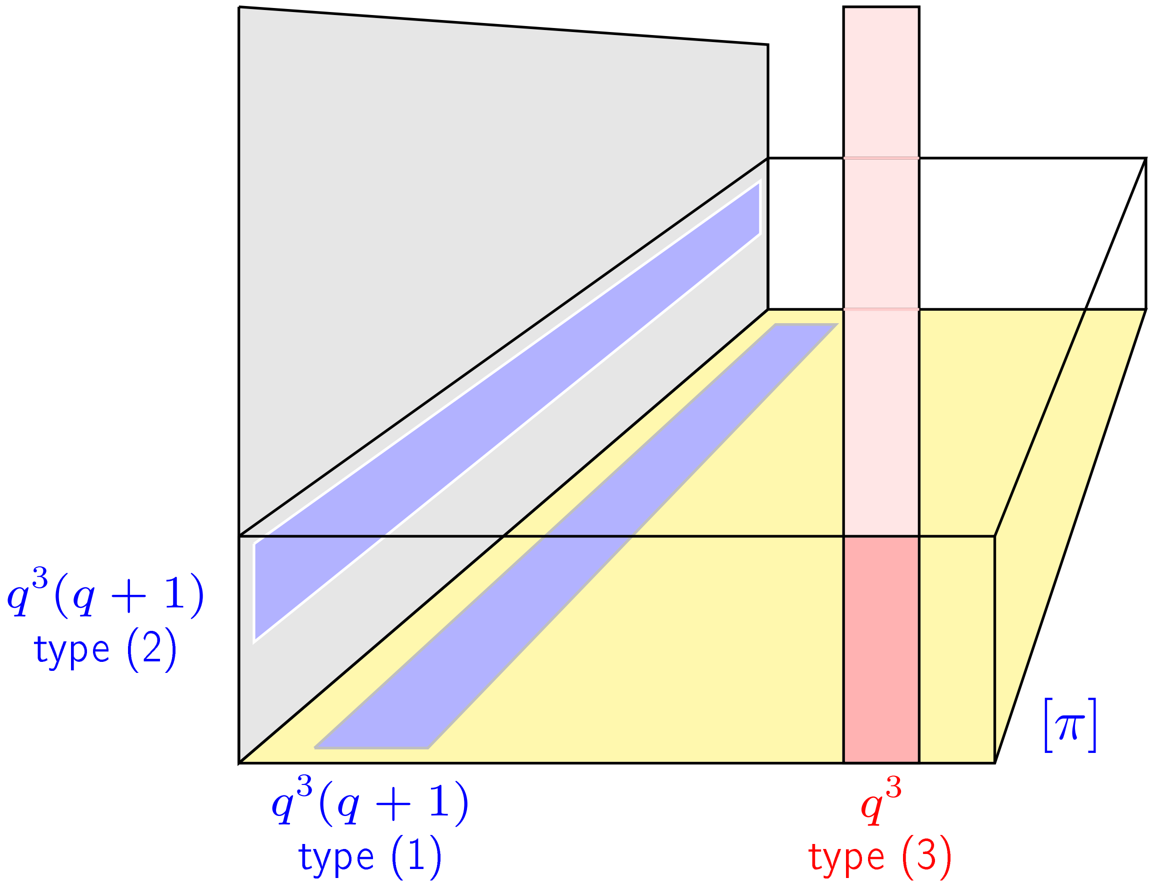
$[\pi]$: 1-neighbour class of planes.

There exist three possibilities for the intersection of a subspace S of shape $(3, 3, 1, 0)$ with $[\pi]$:

(1) S is contained in $[\pi]$ and contained in a plane from $[\pi]$;

(2) S is contained in $[\pi]$ but is not contained in a plane from $[\pi]$;

(3) S is not contained in $[\pi]$.

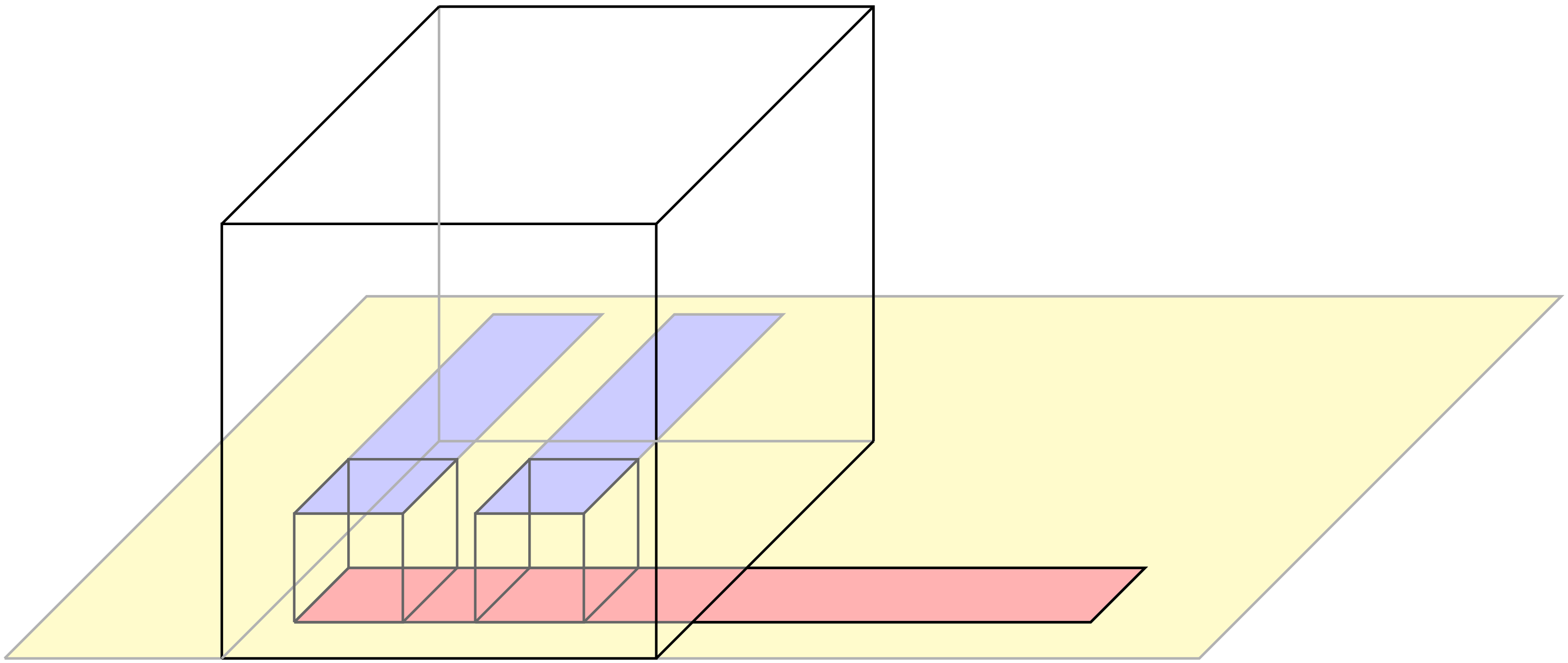

$q^3(q+1)$
type (2)
$q^3(q+1)$
type (1)
q^3
type (3)
$[\pi]$

$A =$ # of subspaces from the spread of type (1) or (2)

$B =$ # of subspaces from the spread of type (3)

$$\left| \begin{array}{rcrcl} A & + & B & = & q^5 + q^3 \\ q^3(q+1)A & + & q^3 B & = & q^6(q^3 + q^2 + q + 1) \end{array} \right.$$

$A = q^3$, $B = q^5$ for every neighbour class of planes $[\pi]$



Observation: There are at least $q^3 - q + 1$ subspaces of type (1) and at most $q - 1$ subspaces of type (2) in any class $[\pi]$.

Count the number of pairs $(S, [\pi])$, where

- $\diamond$ S is a subspace from the spread of type (2).
- $\diamond$ $[\pi]$ is a 1-neighbour class of planes containing π

For each S there exist q choices for π. Therefore

$$\#(S, [\pi]) = q^5 + q^3.$$

On the other hand, for each class $[\pi]$ there exist at most $q - 1$ choices for S. Hence

$$\#(S, [\pi]) \leq (q^3 + q^2 + q + 1)(q - 1) = q^4 - 1,$$

a contradiction.

More generally:

Theorem. Let R be a chain ring of length $m \geq 3$. Let $\Pi = \mathrm{PHG}({}_R R^n)$. There exists no κ-spread of Π for $\kappa = (\underbrace{m, \ldots, m}_{n/2}, \underbrace{m-2, \ldots, m-2}_{n/2-1}, 0)$.

Open problem:

Find a "nice" necessary and sufficient condition on κ for the existence κ-spread in $\mathrm{PHG}({}_R R^n)$.