# ILP techniques for binary subspace codes


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joint work with

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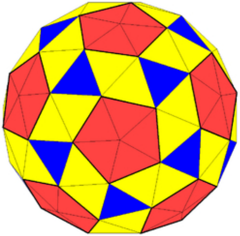

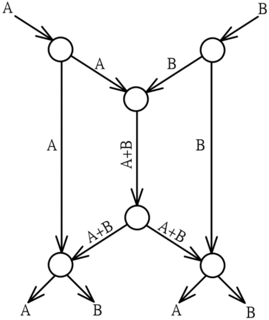


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# Subspace Codes

- codeword: subspace of $\mathbb{F}_q^v$
- distance $d$: graph theoretic distance in the Hasse diagram of the subspace lattice of $\mathbb{F}_q^v$; $U, W \le \mathbb{F}_q^v$:

$$d(U, W) = \dim(U) + \dim(W) - 2\dim(U \cap W)$$

- problem: find a *large* set of subspaces of $\mathbb{F}_q^v$ with pairwise distances $\ge d$
- constant dimension code: all codewords have dimension $k$

# Algorithmic approaches for the determination of the maximum cardinality

**Reference**

Axel Kohnert and S.K. (2008): Construction of large constant dimension codes with a prescribed minimum distance.
In: Calmet, Jacques (Eds.): Mathematical Methods in Computer Science Springer 31-42.

# Algorithmic approaches for the determination of the maximum cardinality

- formulation as a clique problem in a graph (subspaces are vertices, two vertices are connected by an edge iff their distance is $\geq d$) $\rightsquigarrow$ `cliquer` (ask Patric)

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- formulation as a Diophantine equation system (using slack variables) $\rightsquigarrow$ `LLL based solver` (ask Alfred)

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- formulation as a Diophantine equation system (using slack variables) $\rightsquigarrow$ `LLL based solver` (ask Alfred)
- formulation as an Integer Linear Program (chosen subspaces are variables) $\rightsquigarrow$ e.g. `CPLEX`, `Gurobi` (this talk)

**Reference** (sorry, no use of automorphisms here. . . )

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# An example

**Honold, Kiermaier, and K. (2015)**

The maximum cardinality of a constant dimension code with parameters $q = 2$, $v = 6$, $k = 3$, and minimal distance $d = 4$ is 77. There are exactly 5 non-isomorphic extremal codes. Some of these can be generalized to arbitrary $q$.

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Given the corresponding graph $G = (V, E)$ we can formulate

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \le 1 \quad \forall \{u, v\} \in \overline{E} = \binom{V}{2} \backslash E \\ & x_v \in \{0, 1\} \quad \forall v \in V \quad (x_v \in [0, 1] \quad \forall v \in V) \end{aligned}$$

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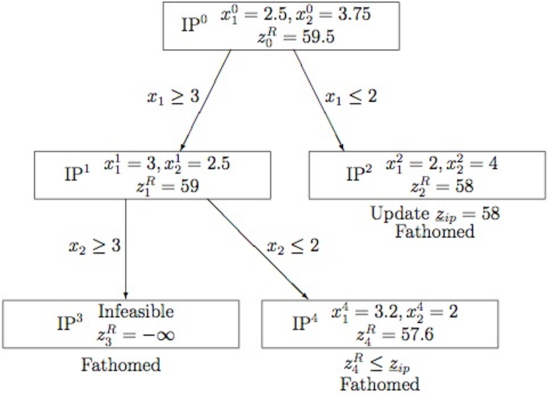
- solution time of the LP-relaxation: 5 seconds $\rightsquigarrow$ 93
- solution time of the ILP: hopeless

# Branch & Bound

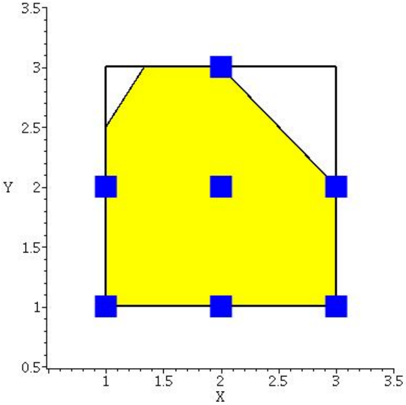
max $\quad 13x_1 + 8x_2$

s.t. $\quad x_1 + 2x_2 \leq 10,\ 5x_1 + 2x_2 \leq 20,\ x_1, x_2 \in \mathbb{Z}_{\geq 0}$



# Polyhedral descriptions

$$2Y - 3X \leq 2$$
$$X + Y \leq 5$$
$$1 \leq X \leq 3$$
$$1 \leq Y \leq 3$$
$$X, Y \in \mathbb{Z}$$



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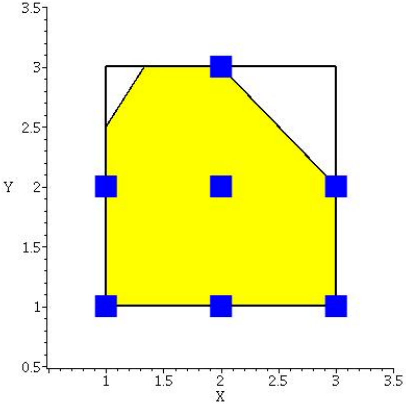



Add the valid constraint $-X + Y \leq 1$

# A perfect polyhedral description

$2Y - 3X \leq 2$

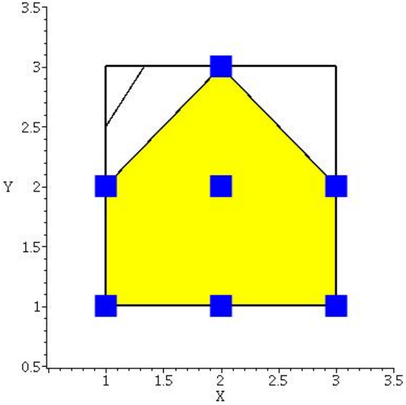
$X + Y \leq 5$

$-X + Y \leq 1$

$1 \leq X \leq 3$

$1 \leq Y \leq 3$

$X, Y \in \mathbb{Z}$




The LP-relaxation coincides with the ILP formulation!

($2Y - 3X \leq 2$ is superfluous.)

# A better polyhedral description for our example

- use independent set constraints:
  $\sum_{v \in I} x_v \leq 1$ for an independent set $I$

$$\sum_{E \leq \mathbb{F}_2^6 \,:\, U \leq E, \dim(E)=3} x_E \leq 1 \quad \forall U \leq \mathbb{F}_2^6 \,:\, \dim(U) = 2$$

  solution time of the LP-relaxation: 3 seconds $\rightsquigarrow$ 93

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- use independent set constraints:
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  solution time of the LP-relaxation: 3 seconds $\rightsquigarrow$ 93
- use sub-graph constraints:
  $\sum_{v \in G'} x_v \leq \alpha(G')$ for a subgraph $G' \leq G$
  $$\sum_{E \leq \mathbb{F}_2^6 \,:\, U \leq E, \dim(E)=3} x_E \leq 9 \quad \forall U \leq \mathbb{F}_2^6 \,:\, \dim(U) = 1$$
  solution time of the LP-relaxation: 1.5 seconds $\rightsquigarrow$ 81

# Symmetry kills the ILP solver...

**Symmetry group**

$GL(6,2)$ of order 20 158 709 760

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## Symmetry group

$GL(6, 2)$ of order 20 158 709 760

## General approach

- identify some appropriate (geometric) sub-configuration
- generate all sub-configurations up to isomorphism
- prescribe each of the sub-configurations (each one separately, Step 1)
- exclude all sub-configurations (Step 2)

# Appropriate sub-configurations

## 9-configurations

A subset of the code consisting of 9 planes passing through a common point $P$.

$$\sum_{E \leq \mathbb{F}_2^6 \,:\, U \leq E,\, \dim(E)=3} x_E = 9 \text{ where } U \leq \mathbb{F}_2^6 \;:\; \dim(U) = 1$$

Up to isomorphism there are just four 9-configurations.

## 17-configurations

A subset of size 17 consisting of two 9-configurations with a common codeword.
There are 12 770 isomorphism types of 17-configurations.

# Another example – the mixed dimensional case

**Problem**

Determine the maximum sizes $A_2(6, d)$ of binary ($q = 2$) "mixed-dimension" subspace codes with packet length $v = 6$ and minimum subspace distance $d \in \{1, 2, 3, 4, 5, 6\}$.

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## Trivial case

- $d = 1$: take all subspaces $\rightsquigarrow$ 2825

## Easy cases (theoretical and computational)

- $d = 2$: take all subspaces of odd dimension $\rightsquigarrow$ 1521; for the upper bound a result of Ahlswede and Aydinian (On error control codes for random network coding, 2009) can be used
- $d = 5, 6$: plane spreads $\rightsquigarrow$ 9

# Valid inequalities for $d = 4$

$$\sum_{E \le \mathbb{F}_2^6 \,:\, U \le E,\, \dim(E)=3} x_E \le 1 \quad \forall U \le \mathbb{F}_2^6 \,:\, \dim(U) = 2$$

$$\sum_{E \le \mathbb{F}_2^6 \,:\, E \le U,\, \dim(E)=3} x_E \le 1 \quad \forall U \le \mathbb{F}_2^6 \,:\, \dim(U) = 4$$

$$\sum_{E \le \mathbb{F}_2^6 \,:\, U \le E,\, \dim(E)\le 5} x_E \le 9 \quad \forall U \le \mathbb{F}_2^6 \,:\, \dim(U) = 1$$

$$\sum_{E \le \mathbb{F}_2^6 \,:\, E \le U,\, \dim(E)\ge 1} x_E \le 9 \quad \forall U \le \mathbb{F}_2^6 \,:\, \dim(U) = 5$$

$$\sum_{P \le U \,:\, \dim(P)=1} x_P + x_U + x_{U'} + \sum_{E \le \mathbb{F}_2^6 \,:\, U \le E,\, \dim(E)=3} x_E \le 1 \quad \forall U \le U' \le \mathbb{F}_2^6 \,:\, \dim(U) = 2,\, \dim(U') = 4$$

$$\ldots$$

Several further inequalities can be stated. In the literature the problem is known as Erdős-Ko-Rado sets.

# Appropriate sub-configurations for $d = 4$

- 9-configurations
- 17-configurations
- *something* that still needs to be discovered… (or just prescribing a few points)

# Excluding 17-configurations

- let $U, U' \leq \mathbb{F}_2^6$, $\dim(U) = \dim(U') = 1$; $\mathcal{C}$ the entire code
- $\sum_{E \leq \mathbb{F}_2^6 \,:\, U \leq E,\, \dim(E)=3} x_E \leq 9$ (affects 155 $x_E$; $S_1$)
- $\sum_{E \leq \mathbb{F}_2^6 \,:\, U' \leq E,\, \dim(E)=3} x_E \leq 9$ (affects 155 $x_E$; $S_2$)
- we consider the three-dimensional subspaces $E$ with either $U \leq E$ or $U' \leq E$ (295 cases; $S_1 \cup S_2$) for $U \neq U'$

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- if $\mathcal{C} \cap S_1 \cap S_2 \neq \emptyset$ then $|\mathcal{C} \cap (S_1 \cup S_2)| \leq 16$ (no 17-configuration)
- if $\mathcal{C} \cap S_1 \cap S_2 = \emptyset$ then $|\mathcal{C} \cap (S_1 \cup S_2)| \leq 18$
- $|\mathcal{C} \cap S_1 \cap S_2| \leq 1$

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- $|\mathcal{C} \cap S_1 \cap S_2| \le 1$

$$\sum_{E \in S_1 \cap S_2} 3x_E \; + \sum_{E \in (S_1 \cup S_2) \setminus (S_1 \cap S_2)} 1x_E \;\le\; 18$$

(Big-$M$ constraint for conditional inequalities.)

# Results for $d = 4$

- prescribe a 17-configuration $\rightsquigarrow$ 77 (only 3-dimensional subspaces are used)
- prescribe a 9-configuration, exclude 17-configurations $\rightsquigarrow$ $\leq$ 74 (even using the LP relaxation)
- exclude 9-configurations $\rightsquigarrow$ 51 ... 98 (another sub-configuration to kill symmetry is needed)

**Summary**

The maximum cardinality for $d = 4$ lies between 77 and 98.

(Of course the exact upper bound is 77 and the already classified 5 isomorphism types are the complete list of extremal codes.)

# Results for $d = 3$

- an example of cardinality 104 has been found
- prescribe a 17-configuration $\rightsquigarrow$ (not completed yet)
- prescribe a 9-configuration, exclude 17-configurations $\rightsquigarrow$ $\leq 97 \ldots 114$ (still too weak)
- exclude 9-configurations $\rightsquigarrow$ $\leq 119$ (still very weak)

**Summary**

The maximum cardinality for $d = 3$ lies between 104 and 119.

(We do not know the exact answer yet. The previously best known bounds were, up to our knowledge, 85 and 123.)

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(We do not know the exact answer yet. The previously best known bounds were, up to our knowledge, 85 and 123.)

Thank you very much for your attention!