To the theory of q-ary Steiner and other-type trades

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ALCOMA 2015, March 15–20, 2015. Kloster Banz

Abstract

- We consider a rather general class of trades, which generalizes several known types of trades, including latin trades, Steiner $(k-1, k, v)$ trades, extended 1-perfect bitrades.
- We prove a characterization of minimal (in the sence of the weight-distribution bound) trades in terms of isometric bipartite distance-regular subgraphs of the original distance-regular graph.
- An an application, we find the minimal cardinality of q-ary Steiner $(k-1, k, v)$ bitrades and show a connection of such bitrades with dual polar subgraphs of the Grassmann graph $Gr_q(v, k)$.

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Outline

- Definitions (distance-regular graphs, eigenfunctions, clique bitrade)
- Clique bitrade: equivalent definitions
- Weight-distribution lower bound.
- Minimal bitrades and generated subgraphs.
- Examples (latin bitrades, Steiner bitrades, binary 1-perfect bitrades)
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Def: Distance-regular graphs

A connected graph Γ is called distance-regular if there are constants b_0, b_1, ..., $b_{\mathrm{diam}(\Gamma)-1}$, c_1, c_2, ..., $c_{\mathrm{diam}(\Gamma)}$ (called intersection numbers) such that for exery vertices x and y at distance i

$$|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i,$$

$$|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i,$$

where $\Gamma_j(x)$ denotes the set of vertices at distance j from x.

Def: eigenfunction, eigenvalues

An eigenfunction of a graph $\Gamma = (V, E)$ is a function $f : V \to \mathbb{R}$ that is not constantly zero and satisfies

$$\sum_{y \in \Gamma_1(x)} f(y) = \theta f(x) \tag{1}$$

for all x from V and some constant θ, which is called an eigenvalue of Γ.

(k, s, m) pairs, Delsarte pairs

- Let Γ be a connected regular graph of degree k. Assume that S is a set of $(s+1)$-cliques in Γ such that every edge of Γ is included in exactly m cliques from S; in this case, we will say that the pair (Γ, S) is a (k, s, m) pair.
- A clique in a distance-regular graph of degree k is called a Delsarte clique if it has exactly $1 - k/\theta$ elements, where θ is the minimal eigenvalue of the graph.
- A (k, s, m) pair (Γ, S) is called a Delsarte pair if Γ is a distance-regular graph and $s = -k/\theta$.

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Def: bitrade

- Let (Γ, S) be a (k, s, m) pair. A couple (T_0, T_1) of mutually disjoint nonempty vertex sets is called an S-bitrade, or a clique bitrade, if every clique from S either intersects with each of T_0 and T_1 in exactly one vertex or does not intersect with both of them (in particular, this means that each of T_0, T_1 is an independent set in Γ).
- A set of vertices T_0 is called an S-trade if there is another set T_1 (known as a mate of T_0) such that the pair (T_0, T_1) is an S-bitrade.
- Note that there are differences in terminology.
We use "bitrade = (trade, trade)"
not "trade = (leg, leg)".

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A bitrade criterion

Theorem

Let (Γ, S) be a (k, s, m) pair. Let $T = (T_0, T_1)$ be a pair of disjoint nonempty independent sets of vertices of Γ. The following assertions are equivalent.

(a) *T is an S-bitrade.*

(b) *The function*

$$f^T(\bar{x}) = \chi_{T_0}(\bar{x}) - \chi_{T_1}(\bar{x}) = \begin{cases} (-1)^i & \text{if } \bar{x} \in T_i,\ i \in \{0,1\} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

is an eigenfunction of Γ with eigenvalue $\theta = -k/s$.

(c) *The subgraph Γ^T of Γ generated by the vertex set $T_0 \cup T_1$ is regular with degree $-\theta = k/s$ (as T_0 and T_1 are independent sets, this subgraph is bipartite).*

Distance-regular graph $\to$ Delsarte pair

Lemma

If, under notation and hypothesis of the previous Theorem, (a)–(c) *hold and, additionally, the graph* Γ *is distance-regular, then*

- θ *is the minimal eigenvalue of* Γ*,*
- $s + 1$ *is the maximal order of a clique in* Γ*, and*
- (Γ, S) *is a Delsarte pair.*

Calculating the weight distribution of an eigenfunction

Lemma

The weight distribution

$$W(x) = \left(\sum_{y \in \Gamma_0(x)} f(y), \sum_{y \in \Gamma_1(x)} f(y), \ \ldots, \sum_{y \in \Gamma_{\mathrm{diam}(\Gamma)}(x)} f(y) \right)$$

of an eigenfunction f of a distance-regular graph Γ is calculated as $(f(x) W^i_{A,\theta})_{i=0}^{\mathrm{diam}(\Gamma)}$ where the coefficients $W^i_{A,\theta}$ are derived from the intersection array $A = (b_0, \ldots, c_{\mathrm{diam}(\Gamma)})$ of Γ and the eigenvalue θ that corresponds to f.

Corollary (the weight-distribution (w.d.) bound)

An eigenfunction f of a distance-regular graph has at least $\sum_{i=0}^{\mathrm{diam}(\Gamma)} |W^i_{A,\theta}|$ nonzeros, in notation of the Lemma.

Trades that meet the w.d. bound

Theorem

Let (Γ, S) be a (k, s, m) Delsarte pair. Let $T = (T_0, T_1)$ be a pair of disjoint nonempty independent sets of vertices of Γ. The following are equivalent.

(a') *T is an S-bitrade meeting the w.d. bound.*

(b') *The function f^T is an eigenfunction of Γ meeting the w.d. bound with eigenvalue $-k/s$.*

(c') *The subgraph Γ^T is a regular isometric subgraph with degree k/s.*

Distance regularity of Γ^T

Theorem

Assume that, under the notation and the hypothesis of the previous Theorem, (a')–(c') *hold. Then the graph Γ^T is distance-regular.*

Corollary

For every distance-regular graph Γ admitting a Delsarte pair, there is a sequence $A' = (b'_0, \ldots, b'_{\mathrm{diam}(\Gamma)-1}; c'_1, \ldots, c'_{\mathrm{diam}(\Gamma)})$ such that the existence of a clique bitrade in Γ meeting the w.d. bound is equivalent to the existence of an isometric distance-regular subgraph with intersection array A'.

Clique designs

Given a Delsarte pair (Γ, S), we define a clique design as a set of vertices that intersects with every clique from S in exactly one vertex. Examples of clique designs: distance-2 MDS codes (Hamming graphs), STS, SQS, ... (Johnson graphs), extended 1-perfect binary codes (halved n-cube), STS_q (Grassmann graph).

Example. Latin bitrades

- The vertex set of the Hamming graph $H(n,q)$ is the set $\{0,\ldots,q-1\}^n$ of words of length n over the alphabet $\{0,\ldots,q-1\}$. Two words are adjacent whenever they differ in exactly one position. The graph $H(n,2)$ is also known as the n-cube, or the hypercube of dimension n.
- The clique designs in Hamming graphs are known as the latin hypercubes (in coding theory, these objects are known as the distance-2 MDS codes), and the clique bitrades, as the latin bitrades [1]. The most studied case, which corresponds to the latin squares, is $n=3$, see e.g. [2].
- The graph corresponding to a minimal bitrade is $H(n,2)$.

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Example. Steiner trades

- The vertices of the Johnson graph $J(n, w)$ are the binary words of length n and weight (the number of ones) w. Two words are adjacent whenever they differ in exactly two positions. The graphs $J(n, w)$ and $J(n, n - w)$ are isomorphic, and below we assume $2w \leq n$.
- The clique designs in Johnson graphs are known as the Steiner $\mathrm{S}(w - 1, w, n)$ systems, and the clique bitrades, as the Steiner $\mathrm{T}(w - 1, w, n)$ bitrades. The subgraph corresponding to a minimal bitrade is $H(w, 2)$; an example of the vertex set of such subgraph is $\{(x, \bar{x}, 0, ..., 0) \mid x, \bar{x} \in \{0, 1\}^w,\ \bar{x} \text{ is opposite to } x\}$. The minimal bitrade cardinality was found in [3].
- In the case $w = 3$, the minimal trade is known as the Pasch configuration, or the quadrilateral.

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Example. Halved hypercube

- The vertices of the halved n-cube are the even-weight binary words of length n (i.e., a part of the bipartite n-cube). Two words are adjacent whenever they differ in exactly two positions.
- A maximal clique is the set of binary n-words adjacent in $H(n, 2)$ to a fixed odd-weight word. The clique designs in halved n-cubes are the extended 1-perfect codes. Such codes exist if and only if n is a power of two.
- The minimal cardinality of a bitrade is $2^{n/2}$. An example of a minimal bitrade is $\{(x, x) \mid x \in \{0, 1\}^{n/2}\}$; bitrades exist if and only if n is even. The graph corresponding to a minimal bitrade is $H(n/2, 2)$.

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q-ary Steiner systems

- Let F_q^n be an n-dimensional vector space over the Galois field F_q of prime-power order q. The Grassmann graph $Gr_q(n,d)$ is defined as follows. The vertices are the d-dimensional subspaces of F_q^n. Two vertices are adjacent whenever they intersect in a $(d-1)$-dimensional subspace.
- All vertices that include a fixed $(d-1)$-dimensional subspace form a clique in $Gr_q(n,d)$; if $n \geq 2d$ then this clique is maximal. We form S from all such cliques.
- A set of vertices that intersect with every cliques from S in exactly one vertex is known as a *q-ary Steiner* $\mathrm{S}_q[d-1,d,n]$ *system*. Constructing q-ary Steiner $\mathrm{S}_q[d-1,d,n]$ systems with $d \geq 3$ is not easy; at the moment, only the existence of $\mathrm{S}_2[2,3,13]$ is known in this field [4].
- An S-bitrade is called a Steiner $\mathrm{T}_q[d-1,d,n]$ bitrade.

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dual polar graph

- The dual polar graph $D_d(q)$ is a subgraph of $Gr_q(2d, d)$ that has as vertices the maximal isotropic subspaces with respect to the quadratic form $Q(v_1, \ldots, v_d, u_1, \ldots u_d) = v_1 u_1 + \cdots + v_d u_d$ (i.e., the subspaces of dimension d on which the form vanishes).
- $D_d(q)$ is a bipartite isometric subgraph of $Gr_q(2d, d)$ and has degree $(q^d - 1)/(q - 1)$ (as required :)).

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Minimal cardinality of a q-ary Steiner bitrade

Theorem

The minimal cardinality of a Steiner $\mathrm{T}_q[d-1, d, n \geq 2d]$ bitrade is

$$\prod_{i=1}^{d}(q^{d-i}+1) = \sum_{i=0}^{d} q^{\binom{i}{2}} \begin{bmatrix} d \\ i \end{bmatrix}_q, \tag{3}$$

which is also the minimal number of nonzeros of an eigenfunction with the minimal eigenvalue in $Gr_q(n, d)$, $n \geq 2d$.

For the proof, it remains to note that $Gr_q(2d, d)$ is an isometric subgraph of $Gr_q(n, d)$.

A small example

- The minimal cardinality of $T_2[2, 3, n]$ is $2 \cdot 15 = 1 + 7 + 14 + 8$.
- Such minimal bitrade can be considered as a q-ary analog of the Pasch configuration.
- Note that the Pasch configuration, together with its trade mate, consists of all eight weight-3 binary words of length 6 on which the form $Q(\ldots) = v_1 u_1 + v_2 u_2 + v_3 u_3$ vanishes.

$\begin{array}{|c|c|c|}\hline v_1 & v_2 & v_3 \\ \hline u_1 & u_2 & u_3 \\ \hline\end{array}$:
$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$
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- Note that the Pasch configuration, together with its trade mate, consists of all eight weight-3 binary words of length 6 on which the form $Q(\ldots) = v_1 u_1 + v_2 u_2 + v_3 u_3$ vanishes.

v_1	v_2	v_3
u_1	u_2	u_3

:

1	1	1
0	0	0

1	0	0
0	1	1

0	1	0
1	0	1

0	0	1
1	1	0

1	1	1
0	0	0

0	1	1
1	0	0

1	0	1
0	1	0

1	1	0
0	0	1

A small example

- The minimal cardinality of $T_2[2,3,n]$ is $2 \cdot 15 = 1 + 7 + 14 + 8$.
- Such minimal bitrade can be considered as a q-ary analog of the Pasch configuration.
- Note that the Pasch configuration, together with its trade mate, consists of all eight weight-3 binary words of length 6 on which the form $Q(...) = v_1 u_1 + v_2 u_2 + v_3 u_3$ vanishes.

$$\begin{array}{|c|c|c|}\hline v_1 & v_2 & v_3 \\ \hline u_1 & u_2 & u_3 \\ \hline\end{array} : \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline\end{array}$$

Conclusion

- We considered trades that can be treated as clique trades in distance-regular graphs.
- Some other types of trades can also be considered as clique trades, but the corresponding graphs are not distance-regular. For example, q-ary 1-perfect trades with $q > 2$, MDS trades with distance > 2, Steiner (t, k, n) trades with $t < k - 1$.