

# Recursive construction of subspace designs

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joint work with Michael Braun, Axel Kohnert and Reinhard Laue

## Definition (block design)

Let  $V$  be a  $v$ -element set.

$D \subseteq \binom{V}{k}$  is a  $t$ - $(v, k, \lambda)$  (block) design

if each  $T \in \binom{V}{t}$  is contained in exactly  $\lambda$  elements of  $D$ .

$q$ -analog in combinatorics:

Replace subset lattice by subspace lattice!

## Definition (subspace design)

Let  $V$  be a  $v$ -dimensional  $\mathbb{F}_q$  vector space.

$D \subseteq \left[ \begin{smallmatrix} V \\ k \end{smallmatrix} \right]_q$  is a  $t$ - $(v, k, \lambda)_q$  (subspace) design

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## Infinite families

Only known infinite nontrivial families with  $t \geq 2$ :

- ▶  $2-(v, 3, q^2 + q + 1)_q$  for  $v \geq 7$ ,  $\gcd(v, 6) = 1$   
(S. Thomas 1987; Suzuki 1990, 1992)
- ▶  $2-(m\ell, 3, q^3 \frac{q^{\ell-5}-1}{q-1})_q$   
for  $m \geq 3$ ,  $\ell \geq 7$  and  $\ell \equiv 5 \pmod{6(q-1)}$   
(T. Itoh 1998)

## Goal

Construction of new infinite families!

## Definition

Fix a parameter set  $t$ - $(v, k, \lambda)_q$ .

A **large set**  $LS_q[N](t, k, v)$

is a partition of  $\begin{bmatrix} V \\ k \end{bmatrix}_q$  into  $N$   $t$ - $(v, k, \lambda)_q$  designs.

## Remarks

- ▶  $\lambda = \begin{bmatrix} v-t \\ k-t \end{bmatrix}_q / N$  is determined by  $N, v, k, t, q$ .
- ▶ Only known nontrivial large sets with  $t \geq 2$ :
  - ▶  $LS_3[2](2, 3, 6)$  (M. Braun 2005)
  - ▶  $LS_2[3](2, 3, 8)$  (M. Braun; A. Kohnert; P. Östergård; A. Wassermann 2014)
  - ▶  $LS_5[2](2, 3, 6)$  (new, computer construction)
- ▶ For large sets of *ordinary* block designs:  
Powerful recursion methods!  
(Khosrovshahi, Ajoodani-Namini 1991)
- ▶ Adjust those recursion methods to subspace designs!

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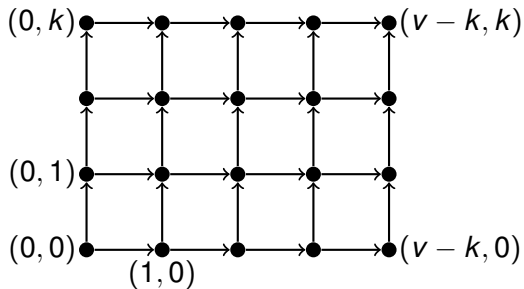
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## Definition (Directed grid graph)





## Bijection

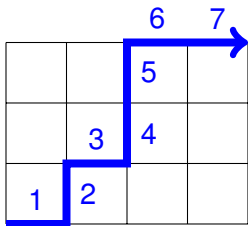
paths from  $(0, 0)$  to  $(v - k, k) \xleftrightarrow{1\text{-to-}1} k\text{-subsets } K \text{ of } V$

vertical step  $\longleftrightarrow$  element in  $K$

horizontal step  $\longleftrightarrow$  element not in  $K$

### Example

$V = \{1, 2, 3, 4, 5, 6, 7\}$ .



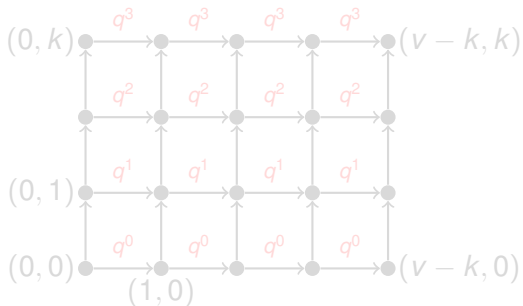
vertical steps: 2, 4, 5  $\rightsquigarrow$   $\{2, 4, 5\} \in \binom{V}{3}$

## Question

Is there a  $q$ -analog of this bijection?

- ▶ Wanted: paths in some graph  $\xleftrightarrow{1\text{-to-1}} [k]_q$
- ▶ As good: paths in some graph  $\xleftrightarrow{1\text{-to-1}} \text{rref in } \mathbb{F}_q^{k \times v}$

Definition (Directed  $q$ -grid multigraph)

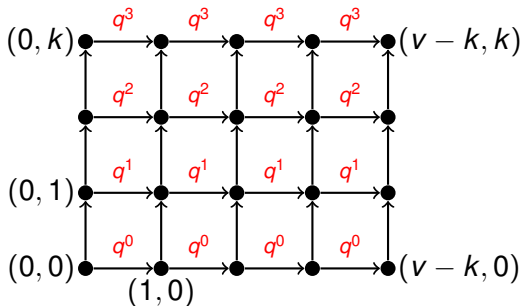


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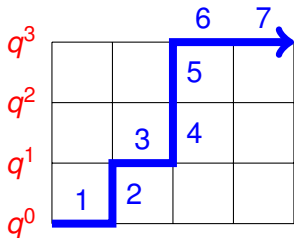
$k$ -subspaces  $K$  of  $V \xleftrightarrow{1\text{-to-1}}$  paths from  $(0, 0)$  to  $(v - k, k)$

vertical step  $\longleftrightarrow$  pivot column

horizontal step  $\longleftrightarrow$  non-pivot column

### Example

$$V = \mathbb{F}_q^7$$



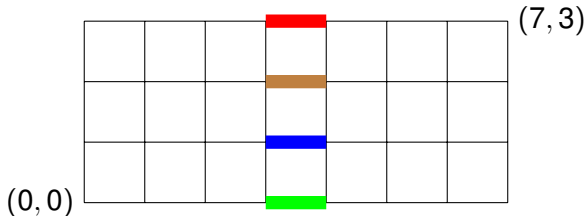
$$\rightsquigarrow \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} q^0 \\ q^1 \\ q^3 \\ q^3 \end{matrix} & \begin{pmatrix} 0 & \mathbf{1} & * & 0 & 0 & * & * \\ 0 & 0 & 0 & \mathbf{1} & 0 & * & * \\ 0 & 0 & 0 & 0 & \mathbf{1} & * & * \end{pmatrix} \end{matrix}$$

## Partitions

Partition of the set of paths from  $(0, 0)$  to  $(v - k, k)$  yields

- ▶ partition of  $\begin{bmatrix} V \\ k \end{bmatrix}_q$
- ▶ identity for Gaussian binomial coefficients
- ▶ ... including bijective proof.
- ▶ New large sets from old ones!

## Example



Partition of paths from (0, 0) to (7, 3) into 4 parts.

► Blue part  $\longleftrightarrow$  
$$\begin{pmatrix} 1 \times 4 \text{ rref} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 2 \times 5 \text{ rref} & \end{pmatrix}$$

► Number of such rref:  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q \cdot q \cdot q^3$

►  $\rightsquigarrow$  identity

$$\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q + q^4 \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + q^8 \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^{12} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = \begin{bmatrix} 10 \\ 3 \end{bmatrix}_q$$

## Example

$$\begin{bmatrix} 10 \\ 3 \end{bmatrix}_q = \begin{bmatrix} 3 \\ 0 \end{bmatrix}_q \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q + q^4 \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q + q^8 \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q + q^{12} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q \begin{bmatrix} 3 \\ 0 \end{bmatrix}_q$$

► Blue part  $\longleftrightarrow$   $\begin{pmatrix} 1 \times 4 \text{ rref} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 2 \times 5 \text{ rref} & \end{pmatrix}$

► Describe this set of subspaces as **join**  $\begin{bmatrix} \mathbb{F}_q^4 \\ 1 \end{bmatrix}_q * \begin{bmatrix} \mathbb{F}_q^5 \\ 2 \end{bmatrix}_q$ .

► Simplified notation:  $\begin{bmatrix} 4 \\ 1 \end{bmatrix} * \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ .

$\rightsquigarrow$  Disjoint union of joins

$$\begin{bmatrix} 10 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} * \begin{bmatrix} 6 \\ 3 \end{bmatrix} \uplus \begin{bmatrix} 4 \\ 1 \end{bmatrix} * \begin{bmatrix} 5 \\ 2 \end{bmatrix} \uplus \begin{bmatrix} 5 \\ 2 \end{bmatrix} * \begin{bmatrix} 4 \\ 1 \end{bmatrix} \uplus \begin{bmatrix} 6 \\ 3 \end{bmatrix} * \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

## Definition

- ▶ Let  $\mathcal{B}_1, \mathcal{B}_2 \subseteq \binom{V}{k}_q$ .
- ▶  $\mathcal{B}_1$  and  $\mathcal{B}_2$   **$t$ -equivalent** if for all  $T \in \binom{V}{t}_q$

$$\#\{\mathbf{B} \in \mathcal{B}_1 \mid T \subseteq \mathbf{B}\} = \#\{\mathbf{B} \in \mathcal{B}_2 \mid T \subseteq \mathbf{B}\}$$

## Example

$q = 1, t = 2, k = 3$ .

$$\mathcal{B}_1 = \{\{1, 2, 4\}, \{1, 3, 6\}, \{2, 5, 6\}, \{3, 4, 5\}\}$$

$$\mathcal{B}_2 = \{\{1, 2, 6\}, \{1, 3, 4\}, \{2, 4, 5\}, \{3, 5, 6\}\}$$

- ▶  $T = \{1, 4\} \implies \lambda(T, \mathcal{B}_1) = \lambda(T, \mathcal{B}_2) = 1$
- ▶  $T = \{2, 3\} \implies \lambda(T, \mathcal{B}_1) = \lambda(T, \mathcal{B}_2) = 0$
- ▶ ... Check all  $T$  ...
- ▶  $\implies \mathcal{B}_1$  and  $\mathcal{B}_2$  are 2-equivalent.



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## Definition

- ▶ Let  $\mathcal{B} \subseteq \binom{V}{k}_q$
- ▶ For  $t \geq 0$ :  $\mathcal{B}$  is  $(N, t)$ -partitionable :  $\iff$   
 $\exists$  partition  $\{\mathcal{B}_1, \dots, \mathcal{B}_N\}$  of  $\mathcal{B}$  s.t. all  $\mathcal{B}_i$  are  $t$ -equivalent.
- ▶ Always:  $\mathcal{B}$  is  $(N, -1)$ -partitionable.

## Lemma

$$\exists \text{LS}_q[N](t, k, v) \iff \binom{v}{k} \text{ is } (N, t)\text{-partitionable.}$$

## Application for $q \in \{3, 5\}$

- ▶  $\exists \text{LS}_q[2](2, 3, 6) \implies \binom{6}{3}$  is  $(2, 2)$ -partitionable
- ▶ *Derived large set*:  $\exists \text{LS}_q[2](1, 2, 5) \implies \binom{5}{2}$  is  $(2, 1)$ -part.
- ▶ *Derived large set*:  $\exists \text{LS}_q[2](0, 1, 4) \implies \binom{4}{1}$  is  $(2, 0)$ -part.

Rules to create  $(N, t)$ -partitionable sets:

### Proposition

The disjoint union of  $(N, t)$ -partitionable sets is  $(N, t)$ -partitionable.

### Basic Lemma

- ▶ Let  $\mathcal{B}_1$  be  $(N, t_1)$ -partitionable
- ▶ Let  $\mathcal{B}_2$  be  $(N, t_2)$ -partitionable

The join  $\mathcal{B}_1 * \mathcal{B}_2$  is  $(N, t_1 + t_2 + 1)$ -partitionable.

## Example

$$\begin{aligned} \begin{bmatrix} 10 \\ 3 \end{bmatrix} &= \\ \begin{bmatrix} 3 \\ 0 \end{bmatrix} * \begin{bmatrix} 6 \\ 3 \end{bmatrix} \uplus \begin{bmatrix} 4 \\ 1 \end{bmatrix} * \begin{bmatrix} 5 \\ 2 \end{bmatrix} \uplus \begin{bmatrix} 5 \\ 2 \end{bmatrix} * \begin{bmatrix} 4 \\ 1 \end{bmatrix} \uplus \begin{bmatrix} 6 \\ 3 \end{bmatrix} * \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ \underbrace{(2, -1) \quad (2, 2)} & \quad \underbrace{(2, 0) \quad (2, 1)} \quad \underbrace{(2, 1) \quad (2, 0)} \quad \underbrace{(2, 2) \quad (2, -1)} \\ \underbrace{(2, -1 + 2 + 1)} & \quad \underbrace{(2, 0 + 1 + 1)} \quad \underbrace{(2, 1 + 0 + 1)} \quad \underbrace{(2, 2 - 1 + 1)} \\ & = (2, 2) \quad = (2, 2) \quad = (2, 2) \quad = (2, 2) \\ & \underbrace{\hspace{15em}} \\ & (2, 2)\text{-partitionable} \end{aligned}$$

$$\implies \begin{bmatrix} 10 \\ 3 \end{bmatrix} \text{ is } (2, 2)\text{-partitionable} \implies \exists \text{LS}_q[2](2, 3, 10)$$

## Example (cont.)

- ▶  $\exists \text{LS}_q[2](2, 3, 10)$  for  $q \in \{3, 5\}$
- ▶  $\implies \exists 2-(10, 3, 1640)_3$  and  $2-(10, 3, 48828)_5$  designs
- ▶ number of blocks: 238247460880 and 208628946735352





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## Theorem

Let  $q \in \{3, 5\}$ .

There exists an  $LS_q[2](2, k, v)$  for all

- ▶  $v \equiv 2 \pmod{4}$  with  $v \geq 6$
- ▶  $k \equiv 3 \pmod{4}$  with  $3 \leq k \leq v - 3$ .

From  $LS_2[3](2, 3, 8)$ :

Theorem? (work in progress!)

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- ▶  $k \equiv 3, 5 \pmod{6}$  with  $3 \leq k \leq v - 3$ .

## Open questions

- ▶ Construct  $LS_q[2](2, 3, 6)$ ,  $q \geq 7$  odd.  
(known for  $q \in \{3, 5\}$ , invariant under Singer<sup>2</sup>)
- ▶ Construct  $LS_2[3](2, 4, 8)$   
(Smallest open case for  $q = 2$ ,  $N = 3$ )
- ▶ When does  $LS_q[N](1, k, v)$  exist? (includes parallelisms)  
Necessary conditions:  $k \mid v$  and  $N \mid \begin{bmatrix} v-1 \\ k-1 \end{bmatrix}_q$   
Z. Baranyai 1975: Sufficient for  $q = 1$ .