Aalto University

Matroid Designs

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Designs

A t-(v, k, λ) design is a collection $\mathcal{B} = \{B_1, \ldots, B_b\}$ of subsets (blocks) of a set X, where

- $v = |X|$
- $k = |B_i|$ for all i
- every subset T with $|T| = t$ is contained in exactly λ blocks

And every t-(v, k, λ) design is also a

$$s\text{-}\left(v, k, \lambda \binom{v-s}{t-s} \binom{k-s}{t-s}^{-1}\right)$$

design for $0 \leq s \leq t$.

q-ary Designs

A t-$(v, k, \lambda; q)$ design over $\mathbb{F}_q$ is a collection $\mathcal{B} = \{B_1, \ldots, B_b\}$ of subspaces (blocks) of the vectorspace $\mathbb{F}_q^v$, where

- $v = \dim \mathbb{F}_q^v$
- $k = \dim B_i$ for all i
- every subspace T with $\dim T = t$ is contained in exactly λ blocks

And every t-$(v, k, \lambda; q)$ design is also a

$$s\text{-}\left(v, k, \lambda \begin{bmatrix} v-s \\ t-s \end{bmatrix}_q \begin{bmatrix} k-s \\ t-s \end{bmatrix}_q^{-1}; q\right)$$

design for $0 \leq s \leq t$.

q-analogues ($q \to 1$)

Sets are "vector spaces" over the field with one element

$$\lim_{q \to 1} \begin{bmatrix} m \\ n \end{bmatrix}_q = \binom{m}{n}$$

So we can explain the similarity between the two kinds of designs as a limit case involving $\mathbb{F}_1$

Matroids

A matroid is a finite set X together with a family $\mathcal{I}$ of subsets of X, such that

i.) $\emptyset \in \mathcal{I}$ or $\mathcal{I} \neq \emptyset$

ii.) For all $A \subseteq B \in \mathcal{I} \;\Rightarrow\; A \in \mathcal{I}$

iii.) If $A, B \in \mathcal{I}$ and $|A| < |B| \;\Rightarrow\;$ there exists $x \in B$ such that $A \cup \{x\} \in \mathcal{I}$

Matroids continued

Furthermore a matroid has:

- A **rank function** $\rho : \mathcal{P}(E) \to \mathbb{N}$
  $\rho(S)$ is the size of a maximal independent subset of S
  think of dimension of span
- Circuits, which are minimal *non-independent* sets
- A **closure operator** $cl : \mathcal{P}(E) \to \mathcal{P}(E)$
  For $S \subseteq E$, $cl(S) := \{x \in E : \rho(A \cup \{x\}) = \rho(A)\}$
  think of span of a set
- **Flats**, which are closed sets
  $cl(F) = F$

Matroids
Examples

- For a finite set X the po-set of subsets forms a matroid, the so called uniform matroid (of rank $|X|$)
- For any subset E of a vector space over a field F take as independent sets the F linearly independent subsets of E, these are called vector matroids
- Another source of matroids are graphs, E is the set of edges and the independent sets are forests, i.e. cycle free subsets of E

Designs on Matroids

Let's consider a t-(v, k, λ) design on a set X of cardinality v.

Definition (A first attempt)

Given the uniform matroid $(X, \mathcal{P}(X))$, a t-(v, k, λ) design is a collection of independent sets of rank k such that all independent sets of rank t are contained in exactly λ of these.

Designs on Matroids

Let's now consider a t-$(v, k, \lambda; q)$ design on $\mathbb{F}_q^v$ of cardinality v.

Definition (A failed attempt)

Given the vector matroid $(\mathbb{F}_q^v, \mathcal{L})$, a t-(v, k, λ) design is a collection of independent sets of rank k such that all independent sets of rank t are contained in exactly λ of these.

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Really?

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \not\subset \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

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Designs on Matroids
For real this time

Definition

Given a matroid $(E, \mathcal{I})$, a t-(v, k, λ) design is a collection of closed sets (flats) of rank k such that all closed sets (flats) of rank t are contained in exactly λ of these.

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- and for the usual designs on sets on the uniform matroid, because independent sets=flats

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- This works for the q-ary case on vector matroids
- and for the usual designs on sets on the uniform matroid, because independent sets=flats
- This definition therefore unifies both types of designs!

$t \Rightarrow t - 1$

What about the "inheritance" property of designs? We would like the following:

Theorem

Let $(E, \mathcal{I})$ be a matroid and D a collection of flats of rank k that form a t-(v, k, λ) design. Then D is also a $t-1$-(v, k, λ_{t-1}) design.

$t \Rightarrow t-1$ trying to prove it

Proof.

Fix a rank $t-1$ flat S and double count the pairs (x, B) where $x \notin S$ and $S \cup \{x\} \subseteq B \in D$.

- We can choose $|E| - |S|$ such x and the flat $cl(S \cup \{x\})$ is contained in λ blocks B, because D is a design.
- Assume S is contained in μ blocks B, then for each such block we can choose $|B| - |S|$ different elements x.

Hence $(|E| - |S|)\lambda = \mu(|B| - |S|)$. Therefore the number of blocks that contain S is given as

$$\mu = \frac{|E| - |S|}{|B| - |S|}\lambda$$

□

What now?

- We actually want μ to be independent of the choice of S.
- But the expression

$$\mu = \frac{|E| - |S|}{|B| - |S|}\lambda$$

depends heavily on the size of S (which might be independent of the rank!) and the sizes of the blocks it is contained in (which might vary as well!).

What now?

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  depends heavily on the size of *S* (which might be independent of the rank!) and the sizes of the blocks it is contained in (which might vary as well!).
- Solution: Turn this problem into a definition!
  or simply: *Das ist VERBOTEN!*

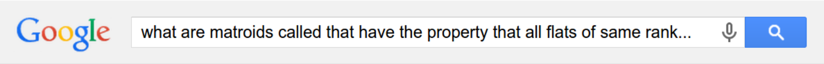
The right Definition

Definition

A matroid allows to define designs in a useful way, if the size of its flats depends only on the rank.

Natural questions:

- What kind of matroids fulfill this condition?
- Is this new?



Perfect Matroid Designs

Definition

A matroid is called a **perfect matroid design**, if all flats of the same rank have the same cardinality.

- Have been christened and studied in the 70's by Peyton Young and Jack Edmonds.
- Except for the uniform and the vector matroids described earlier, only one other family (based on commutative Moufang loops of exponent 3) is known.

What's next?

- Read!
- Young and Edmonds used these matroids to construct new designs from known ones.
- Newer results in matroid theory or raw computational force might help to extend their findings.
- Other (less restricted versions of) designs can be described and studied, since not all sets might appear as flats of rank t.
- Even if no new designs pop up, it's a very nice generalization that unifies designs and their q-ary counterparts while avoiding the 'field with one element'

Thank You!