Construction of block designs admitting a solvable automorphism group

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SVEUČILIŠTE U RIJECI
ODJEL ZA MATEMATIKU

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Abstract

Generalization and refinement of some algorithms for construction of block designs:

- Breadth-first search algorithm for construction of orbit matrices of block designs with a presumed automorphism group, which is a generalization of the algorithm developed by V. Ćepulić.[1]
- Refinement of the obtained orbit matrices for the normal subgroups from some composition series of a solvable automorphism group acting on a block design

[1] V. Ćepulić, On Symmetric Block Designs (40,13,4) with Automorphisms of Order 5, Discrete Math. 128(1-3), 45–60 (1994).

Outline of Talk

Introduction

Tactical decomposition

Algorithm for construction of orbit matrices

Algorithm for refinement of orbit matrices

Results

Introduction - block designs

A t-(v, k, λ) **design** is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties: $|\mathcal{P}| = v$, every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$, and every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$. The elements of $\mathcal{P}$ are called points and the elements of $\mathcal{B}$ are called blocks.
A 2-(v, k, λ) design is called a **block design**.

- Every point of 2-(v, k, λ) design is contained in $r = \frac{v-1}{k-1}\lambda$ blocks and the number of blocks equals to $b = \frac{v(v-1)}{k(k-1)}\lambda$.
- If $|\mathcal{P}| = |\mathcal{B}|$ and $2 \leq k \leq v - 2$, then design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a **symmetric design**.
- isomorphic designs; $Aut(\mathcal{D})$ the full automorphism group of design $\mathcal{D}$

Theorem

Let $M = [m_{ij}]$ and $M' = [m'_{ij}]$ be $v \times b$ incidence matrices of two designs. These designs are isomorphic if and only if there exists a permutation α of $\{1, ..., v\}$ and a permutation β of $\{1, ..., b\}$ such that $m'_{ij} = m_{\alpha(i)\beta(j)}$, $1 \leq i \leq v$, $1 \leq j \leq b$.

Introduction - Group action on a set

- Let a group G act on a non-empty set Ω. For each element $x \in \Omega$ its G-orbit is $xG = \{xg \mid g \in G\}$ and its stabilizer in G is $G_x = \{g \in G \mid xg = x\}$.
- If a finite group G acts on a finite set Ω then the orbit-stabilizer theorem, together with Lagrange's theorem, gives $|xG| = [G : G_x] = |G|/|G_x|$, $\forall x \in \Omega$.
- Let x and y be two elements in Ω, and let $g \in G$ be a group element such that $y = xg$. Then the two stabilizer groups G_x and G_y are related by $G_y = g^{-1} G_x g = G_x^g$.

Example: When a group G acts on itself by conjugation, then $G_a = \{g \mid a^g = g^{-1}ag = a\} = \{g \mid ag = ga\}$ is the centralizer of $a \in G$, denoted by $C_G(a)$. $C_G(A) = \{g \mid a^g = a, \forall a \in A\}$ is the centralizer of $A \subseteq G$.
Example: When G acts on its subgroups by conjugation, then $G_A = \{g \mid A^g = g^{-1}Ag = A\}$ is the normalizer of $A \leq G$, denoted by $N_G(A)$ and the fixed elements A are the normal subgroups of G, denoted by $A \lhd G$.

Theorem

Let group G act on a finite non-empty set Ω and let $H \lhd G$. Further, let x and y be elements of the same G-orbit. Then $|xH| = |yH|$ and a group G/H acts transitively on the set $\{x_i H \mid i = 1, 2, \ldots, h\}$, where $xG = \bigsqcup_{i=1}^{h} x_i H$.

Classes of finite groups:
cyclic < abelian < nilpotent < supersolvable < polycyclic < **solvable** < finitely generated group

- A *subnormal series* of a group G is a sequence of subgroups, each a normal subgroup of the next one. In a standard notation: $\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$. The number $n \in \mathbb{N}$ is called the length of the series.
- A composition series of a group G is a subnormal series of G such that each factor group G_{i+1}/G is a nontrivial simple group. The factor groups are called composition factors.
- A finite group G is called a solvable group if it has a composition series all of whose factors are cyclic groups of prime orders.

Examples: Composition series of $Z_6 \cong \langle \rho \rangle$: $\{1\} \lhd \left\langle \rho^3 \right\rangle \lhd Z_6$, $\{1\} \lhd \left\langle \rho^2 \right\rangle \lhd Z_6$.
Composition series of $Z_{105} \cong \langle a \rangle$: $\{1\} \lhd \left\langle a^{21} \right\rangle \lhd \left\langle a^7 \right\rangle \lhd Z_{105}$.

Tactical decomposition (P. Dembowski, 1958)

- Let M be a $v \times b$ incidence matrix of a block design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$. A *decomposition* of M is any partition $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_m$ of the rows of M and a partition $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$ of the columns of M (M is split into submatrices M_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$).
- We say that the decomposition of the matrix M is *tactical* if the coefficients
  $a_{ij} = |x \in \mathcal{B}_j \mid P\mathcal{I}x|,$ for $P \in \mathcal{P}_i$ arbitrarily chosen,
  $b_{ij} = |P \in \mathcal{P}_i \mid P\mathcal{I}x|,$ for $x \in \mathcal{B}_j$ arbitrarily chosen are well defined.
- The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are called *"condensed forms"* of M, tactical decomposition matrices or point and block orbit matrices, respectively. If an incidence matrix M of a block design has tactical decomposition, then we say that design $\mathcal{D}$ has *tactical decomposition*.
- The action of $G \leq Aut(\mathcal{D})$ induces a tactical decomposition of design $\mathcal{D}$. The orbit lengths distributions we denote by $\nu = (\nu_1, \ldots, \nu_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ for point set and block set of $\mathcal{D}$, respectively.

Example: tactical decomposition

The $(v \times b)$ incidence matrix M of 2-$(8,4,3)$ design

$$M = \left[\begin{array}{c|c|cccc|cc|cccc|cc}
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}\right]$$

$$A = [a_{ij}] = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 \end{bmatrix} \qquad B = [b_{ij}] = \begin{bmatrix} 4 & 0 & 2 & 2 & 2 & 2 \\ 0 & 4 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Point orbit matrix

For the coefficients a_{ij} the following equalities hold:

1) $0 \leq a_{i,j} \leq \beta_j, \quad 1 \leq i \leq m,\ 1 \leq j \leq n,$
2) $\displaystyle\sum_{j=1}^{n} a_{i,j} = r, \quad 1 \leq i \leq m, \quad r = \frac{v-1}{k-1}\lambda,$
3) $\displaystyle\sum_{i=1}^{m} \frac{\nu_i}{\beta_j} a_{i,j} = k, \quad 1 \leq j \leq n,$
4) $\displaystyle\sum_{j=1}^{n} \frac{\nu_t}{\beta_j} a_{s,j} a_{t,j} = \begin{cases} \lambda\nu_t, & s \neq t, \\ \lambda(\nu_t - 1) + r, & s = t. \end{cases}$
5) If $\mathcal{D}$ is a symmetric design the following holds:
$$\sum_{i=1}^{m} \frac{\nu_i}{\beta_s} a_{i,s} a_{i,t} = \lambda\beta_t + \delta_{st}(r - \lambda).$$

Each matrix A of type $m \times n$ whose elements satisfy the propetries 1)-4) is called a **point orbit matrix** for parameters v, k, λ and vectors $\nu = (\nu_1, \ldots, \nu_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$.

Construction of block designs using tactical decomposition consists of two basic steps (Z. Janko, 1992[2]):

1. Construction of orbit matrices for the given automorphism group and parameters of design,
2. Construction of block designs for the obtained orbit matrices. This step is often called an indexing of orbit matrices.

- Because of the large number of possibilities it is often necessary to involve a computer program in both steps of the construction.
- Problem with indexing! One solution is in refinement of orbit matrices for an action of a proposed automorphism group of non-prime order on a block design.

[2] Z. Janko, Coset enumeration in groups and constructions of symmetric designs, Combinatorics '90 (Gaeta, 1990), Ann. Discrete Math. 52 (1992), 275–277.

Algorithm for construction of orbit matrices

- In 1994, V. Ćepulić[3] developed the breadth-first search algorithm (FIFO) for the layer-by-layer construction of all nonisomorphic block orbit matrices for admissible parameters of a symmetric block design with a proposed automorphism group.
- That algorithm was generalized for the layer-by-layer construction of mutually nonisomorphic point orbit matrices for admissible parameters of block designs with their proposed automorphism group, as a part of PhD thesis.[4]

[3] V. Ćepulić, On Symmetric Block Designs (40,13,4) with Automorphisms of Order 5, Discrete Math. 128(1-3), 45–60 (1994)

[4] D. Dumičić Danilović, Generalization and refinement of some algorithms for construction and substructures investigation of block designs, PhD thesis, Zagreb 2014

Algorithm for construction of orbit matrices - reduction

Definition

Let $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}, I_1)$ *and* $\mathcal{D}_2 = (\mathcal{P}, \mathcal{B}, I_2)$ *be block designs and* $G \leq Aut(\mathcal{D}_1) \cap Aut(\mathcal{D}_2) \leq S \equiv S(\mathcal{P}) \times S(\mathcal{B})$. *An isomorphism* α *from* $\mathcal{D}_1$ *onto* $\mathcal{D}_2$ *is called a* G*-isomorphism from* $\mathcal{D}_1$ *onto* $\mathcal{D}_2$ *if there is an automorphism* $\tau : G \to G$ *such that for each* $P, Q \in \mathcal{P}$ *and each* $g \in G$:

$$(P\alpha)(g\tau) = Q\alpha \Leftrightarrow Pg = Q.$$

If $I_1 = I_2 \subseteq \mathcal{P} \times \mathcal{B}$, α *is called a* G*-automorphism of* $\mathcal{D}_1 = \mathcal{D}_2$.

Lema

Let $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}, I_1)$ *and* $\mathcal{D}_2 = (\mathcal{P}, \mathcal{B}, I_2)$ *be block designs, and* $G \leq Aut(\mathcal{D}_1) \cap Aut(\mathcal{D}_2) \leq S \equiv S(\mathcal{P}) \times S(\mathcal{B})$. *A permutation* $\alpha \in S$ *is a* G*-isomorphism from* $\mathcal{D}_1$ *onto* $\mathcal{D}_2$ *if and only if* α *is an isomorphism from* $\mathcal{D}_1$ *onto* $\mathcal{D}_2$ *and* $\alpha \in N_S(G)$.

Theorem

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a block design, $G \leq Aut(\mathcal{D})$, and let the $(m \times n)$-matrix A be an orbit matrix of the design $\mathcal{D}$ with respect to the group G. Further, let $g = (\alpha, \beta)$ be an element of $S = S_m \times S_n$ with the following properties:

1. *if $\alpha(s) = t$, then the stabilizer G_{P_s} is conjugate to G_{P_t}, where $P_s, P_t \in \mathcal{P}$, $\mathcal{P}_s = P_s G$ and $\mathcal{P}_t = P_t G$,*
2. *if $\beta(i) = j$, then G_{x_i} is conjugate to G_{x_j}, where $x_i, x_j \in \mathcal{B}$, $\mathcal{B}_i = x_i G$, $\mathcal{B}_j = x_j G$.*

Then there exists a permutation $g^* \in C_{S(\mathcal{P}) \times S(\mathcal{B})}(G)$, such that

$$\alpha(s) = t \text{ if and only if } g^*(\mathcal{P}_s) = \mathcal{P}_t, \text{ and}$$

$$\beta(i) = j \text{ if and only if } g^*(\mathcal{B}_i) = \mathcal{B}_j.$$

During the construction of orbit matrices, for the reduction we use all permutations from $S_m \times S_n$ which satisfy conditions in the previous theorem; these permutations are defined by vector $\kappa\nu = (\kappa\nu_1, \ldots, \kappa\nu_m)$ and $\kappa\beta = (\kappa\beta_1, \ldots, \kappa\beta_n)$

Definition

Let $\Delta = (\gamma_{ir})$ be an orbit matrix of a (v, k, λ) block design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ with respect to the group $G \leq Aut(\mathcal{D})$, and ν_i, β_j, $1 \leq i \leq m$, $1 \leq j \leq n$, G-orbit lengths of points $\mathcal{P}_i$ and blocks $\mathcal{B}_j$, respectively. A mapping $g = (\alpha, \beta) \in S_m \times S_n$ is called an isomorphism from Δ onto $\Delta' = \Delta g$ if the following conditions hold:

1. *if $\alpha(s) = t$, then the stabilizer G_{P_s} is conjugate to G_{P_t}, where $\mathcal{P}_s = P_s G,\ \mathcal{P}_t = P_t G$;*
2. *if $\beta(u) = v$, then G_{x_u} is conjugate to G_{x_v}, where $\mathcal{B}_u = x_u G,\ \mathcal{B}_v = x_v G$.*

We say that the orbit matrices Δ and Δ' are isomorphic. If $\Delta g = \Delta$, g is called an automorphism of the orbit matrix Δ. All automorphisms of an orbit matrix Δ form the full automorphism group of Δ, denoted by $Aut(\Delta)$.

Refinement of orbit matrices

- Refinement of orbit matrix - what has been done?
- D. Crnković and S. Rukavina in the paper[5] described the algorithm for refinement of orbit matrices of block designs using a principal series of an abelian automorphism group $G = C_1 \times C_2 \times \ldots C_s$:

  $$\{1\} \lhd C_1 \lhd C_1 \times C_2 \ldots C_1 \times C_2 \times \ldots \times C_{s-1} \lhd G.$$

  But, until now, the computer program for the algorithm has not been developed yet.
- The algorithm for the refinement of orbit matrices using a composition series of a proposed solvable group G acting on a block design as its automorphism group;

  $$\{1\} = G_0 \unlhd G_1 \unlhd G_2 \unlhd \ldots \unlhd G_{n-1} \unlhd G_n = G.$$

- Advantage/weakness of the algorithm for refinement

[5] D. Crnković, S. Rukavina, Construction of block designs admitting an Abelian automorphism group, Metrika, 02/2005; 62(2), 175–183

Example: 2-$(36, 15, 6)$ design $\mathcal{D}$ and $Frob_{21} \leq Aut(\mathcal{D})$;
$\nu = \beta = (1, 7, 7, 21)$, $\kappa\nu = \kappa\beta = (0, 1, 1, 2)$

$Frob_{21}$	1	7	7	21
1	1	7	7	0
7	1	4	1	9
7	1	1	4	9
21	0	3	3	9

Z_7	1	7	7	7	7	7
1	1	7	7	0	0	0
7	1	4	1	3	3	3
7	1	1	4	3	3	3
7	0	3	3	4	4	1
7	0	3	3	1	4	4
7	0	3	3	4	1	4

Z_7	1	7	7	7	7	7
1	1	7	7	0	0	0
7	1	4	1	3	3	3
7	1	1	4	3	3	3
7	0	3	3	5	2	2
7	0	3	3	2	5	2
7	0	3	3	2	2	5

Reduction in the refinement of orbit matrices

Let A be an orbit matrix obtained in the $(i-1)$-th iteration of the algorithm for the refinement. In the next one, the i-th iteration, for the reduction we use the following:

- the automorphisms $Aut(A)$, since each automorphism of an orbit matrix determines an G-isomorphism, and
- the isomorphisms, i.e. the permutations of the rows and columns of the resulting orbit matrices obtained by refinement of A, corresponding to the elements of the normalizer of the composition factor $G_{n-i+1}/G_{n-i} \cong Z_p$ in the symmetric group S_p, for some prime number p.

For the construction of designs we have used our own computer programs written in GAP. Limitations of our program are the limitations in the memory of GAP.

After the block designs are constructed, they need to be checked for isomorphism, which we conduct by GAP and its package Design.

Classification of 2-(45,12,3) designs with an involutory automorphism

Up to isomorphism there exists 682 *orbit matrices of a symmetric* 2-$(45, 12, 3)$ *design with an involutory automorphism.*

# of fixed points	5	7	9	11	13	15
# of orbit matrices	233	397	32	4	11	5

# of fixed points	5	7	9	11	13	15
# of designs	603	1898	524	0	225	28

Up to isomorphism there exist 2987 *symmetric* 2-$(45, 12, 3)$ *designs an involutory automorphism.*

$\|\mathrm{Aut}(D)\|$	Structure of $\mathrm{Aut}(D)$	# of designs	$\|\mathrm{Aut}(D)\|$	Structure of $\mathrm{Aut}(D)$	# of designs
51840	$PSp(4,3):Z_2$	1	48	$(Z_3 \times Q_8):Z_2$	3
19440	$(E_{81}:SL(2,5)):Z_2$	1	48	$(Z_4 \times Z_4):Z_3$	1
1296	$E_{27}:(S_4 \times Z_2)$	1	48	$Z_2 \times S_4$	1
486	$E_{81}:Z_6$	1	36	$S_3 \times S_3$	4
486	$E_{81}:S_3$	1	36	$E_9:Z_4$	1
432	$((S_3 \times S_3):Z_2) \times S_3$	2	36	$Z_2 \times (E_9:Z_2)$	1
360	$(Z_{15} \times Z_3):Z_8$	1	32	$Z_4:Q_8$	1
324	$(E_{27}:Z_3):E_4$	1	30	$Z_5 \times S_3$	1
324	$(E_{27}:Z_3):E_4$	1	24	$Z_3 \times Q_8$	1
216	$(Z_3 \times S_3 \times S_3):Z_2$	3	20	$Z_5:Z_4$	1
216	$(E_9:Z_4) \times S_3$	2	20	$Z_5:Z_4$	1
216	$S_3 \times S_3 \times S_3$	2	18	$Z_3 \times S_3$	87
192	$(E_4 \times Q_8):S_3$	2	18	$Z_6 \times Z_3$	4
162	$E_{27}:Z_6$	8	18	$E_9:Z_2$	4
162	$E_{27}:S_3$	5	16	QD_{16}	7
162	$E_{27}:S_3$	1	16	$Z_2 \times D_8$	2
162	$S_3 \times (E_9:Z_3)$	1	16	$(Z_4 \times Z_2):Z_2$	1
144	$(E_9:Z_8):Z_2$	2	12	D_{12}	65
108	$S_3 \times (E_9:Z_2)$	4	8	D_8	12
108	$E_{27}:Z_4$	2	8	Q_8	7
108	$E_{27}:E_4$	1	8	Z_8	4
108	$Z_3 \times S_3 \times S_3$	1	8	E_8	2
64	$(E_4.(Z_4 \times Z_2)):Z_2$	1	8	$Z_4 \times Z_2$	2
64	$((Z_2 \times Q_8):Z_2):Z_2$	1	6	S_3	446
54	$E_{27}:Z_2$	24	6	Z_6	104
54	$E_{27}:Z_2$	9	4	E_4	128
54	$(Z_9:Z_3):Z_2$	6	4	Z_4	71
54	$E_9 \times S_3$	6	2	Z_2	1931
54	$E_9:Z_6$	3			

2-(45,12,3) designs with involutive automorphisms

Classification of 2-(45,5,1) designs with group Z_6

The group $G = \langle \alpha \mid \alpha^6 = 1 \rangle$ may act on a 2-$(45, 5, 1)$ design $\mathcal{D}$ with one of the following four point and block orbit lengths distributions ν and β:

1. $\nu = (4 \times 1, 1 \times 2, 3 \times 3, 5 \times 6)$
   $\beta = (4 \times 1, 1 \times 2, 5 \times 3, 13 \times 6)$,
2. $\nu = (2 \times 1, 2 \times 2, 1 \times 3, 6 \times 6)$
   $\beta = (2 \times 1, 2 \times 2, 3 \times 3, 14 \times 6)$,
3. $\nu = (1 \times 1, 1 \times 2, 2 \times 3, 6 \times 6)$
   $\beta = (1 \times 1, 1 \times 2, 4 \times 3, 14 \times 6)$,
4. $\nu = (1 \times 1, 1 \times 2, 4 \times 3, 5 \times 6)$
   $\beta = (1 \times 1, 1 \times 2, 6 \times 3, 13 \times 6)$.

Up to isomorphism there are exactly 4355 orbit matrices for 2-$(45, 5, 1)$ design $\mathcal{D}$ with the automorphism group $\langle \alpha \mid \alpha^6 = 1 \rangle$.

Distribution	1	2	3	4
# of orbit matrices	20	1066	2798	471

Distribution	1	2	3	4
# of orbit matrices after refinement	0	5843	1400	92

Tablica: Number of orbit matrices for the action of $\langle \alpha^3 \rangle \lhd G \leq Aut(\mathcal{D})$

Distribution	1	2	3	4
# of orbit matrices after refinement	96	28	1262	85

Tablica: Number of orbit matrices for the action of $\langle \alpha^2 \rangle \lhd G \leq Aut(\mathcal{D})$

Distribution	1	2	3	4
# of obtained designs	0	0	0	3
# of nonisomorphic designs	0	0	0	3

Classification of 2-(45,5,1) designs with automorphism group $Z_3 \times Z_3$

Let $\mathcal{D}$ be a 2-$(45, 5, 1)$ design. The group $G \leq Aut(\mathcal{D})$, $G \cong Z_3 \times Z_3$ acts on $\mathcal{D}$ with the following orbit lengths distribution:

$$\nu = (5 \times 9),\ \kappa\nu = (5 \times 0),$$

$$\beta = (11 \times 9),\ \ \kappa\beta = (11 \times 0).$$

# of orbit matrices	51
# of obtained orbit matrices after refinement	5330
# of designs after indexing	39
# of nonisomorphic designs	7

One of the 7 constructed designs has $Aut(\mathcal{D}) \cong (Z_{15} \times Z_3) : Q_8$ of order 360, while the another 6 designs has $Aut(\mathcal{D}) \cong (Z_3 \times Z_3) : Q_8$ of order 72.

Construction of 2-(45,5,1) designs with group S_3

$G = \left\langle \rho, \phi \mid \phi^3 = 1,\ \rho^2 = 1,\ \rho\phi\rho = \phi^{-1} \right\rangle \cong Z_3 : Z_2$. Here, for the refinement we use the following composition series of G: $\{1\} \lhd \langle\phi\rangle \lhd G$. The group S_3 may act on a 2-$(45, 5, 1)$ design with one of the following three orbit lenghts distributions:

1. $\nu = (5 \times 3, 5 \times 6), \quad \beta = (11 \times 3, 11 \times 6)$,
2. $\nu = (13 \times 3, 1 \times 6), \quad \beta = (19 \times 3, 7 \times 6)$,
3. $\nu = (1 \times 1, 1 \times 2, 12 \times 3, 1 \times 6), \quad \beta = (1 \times 1, 1 \times 2, 18 \times 3, 7 \times 6)$.

There are at least 9, up to isomorphism, 2-$(45, 5, 1)$ designs with automorphism group S_3. Seven of them are isomorphic to the ones constructed with the group $Z_3 \times Z_3$, while the two of them have the full automorphism group isomorphic to S_3. The results obtained for the first distribution:

# of orbit matrices	13214
# of orbit matrices after refinement	32861
# of constructed designs	21
# of nonisomorphic designs	9

Classification of 2-(78,22,6) designs with automorphism group $Frob_{39} \times Z_2$

The group

$$G = \left\langle \rho, \sigma, \mu \mid \rho^{13} = \sigma^3 = \mu^2 = 1,\ \rho^\sigma = \rho^3, \rho^\mu = \rho, \sigma^\mu = \sigma \right\rangle \cong Frob_{39} \times Z_2$$

acts on a symmetric 2-$(78, 22, 6)$ design $\mathcal{D}$ as its automorphism group with the following orbit lengths distribution $\nu = \beta = (13, 13, 26, 26)$.

OM	13	13	26	26
13	7	3	6	6
13	3	7	6	6
26	3	3	10	6
26	3	3	6	10

The refinement of OM is based on the composition series:
$\{1\} \lhd \langle\mu\rangle \lhd \langle\rho,\mu\rangle \lhd G$

# of orbit matrices	1
# of orbit matrices for $\langle\rho,\mu\rangle$	1
# of orbit matrices for $\langle\mu\rangle$	4
# of obtained designs	2
# of nonisomorphic designs	1

Theorem

Up to isomorphism, there is one 2-$(78, 22, 6)$ *design* $\mathcal{D}$ *with automorphism group* $G \cong Frob_{39} \times Z_2$. *The group* G *acts semi-standardly on* $\mathcal{D}$ *with point and block orbit lengths distribution* $(13, 13, 26, 26)$ *and it holds that* $G \cong Aut(\mathcal{D})$.

Theorem

There is no $(78, 22, 6)$ *difference set in the group* $Frob_{39} \times Z_2$.

Thank you for your attention!