Complete (k, 3)-arcs from quartic curves

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ALCOMA 2015

Kloster Banz, March 15 - 20, 2015

Outline

- (n, r)-arcs and Coding Theory

- Algebraic constructions of small complete $(n, 3)$-arcs

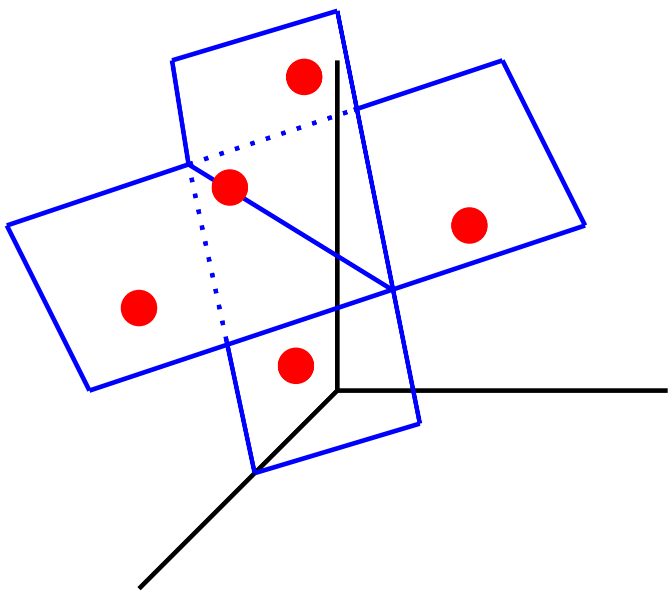
- Possible developments

Complete arcs

Definition (Arc)

$\mathcal{A} \subset AG(r,q), PG(r,q)$ n-arc $\iff$ n points no $r+1$ of which are in a hyperplane

$\mathcal{A}$ complete $\iff$ $\mathcal{A} \not\subset \mathcal{A}'$ $\mathcal{A}'$ $(n+1)$-arc

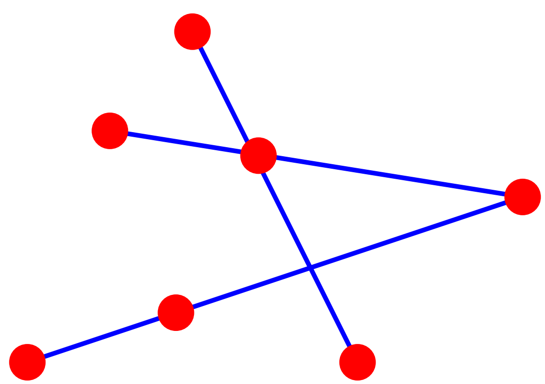


Complete (n, m)-arcs in projective planes

Definition ((n, m)-arc)

$\mathcal{A} \subset AG(2, q), PG(2, q)$
(n, m)-arc
$\iff$
n points
no $m + 1$ of which
are collinear

$\mathcal{A}$
complete
$\iff$
$\nexists$ $\mathcal{A}'$: $\mathcal{A} \not\subset \mathcal{A}'$, $\mathcal{A}'$ $(n + 1, m)$-arc



MDS codes

Linear code $\mathcal{C} < \mathbb{F}_q^N$ d Hamming distance

Singleton Bound

$$[n, k, d]_q \implies d \leq n - k + 1$$

Definition (MDS Codes)

$d = n - k + 1 \implies$ *Maximum Distance Separable* (*MDS*)

MDS $[n, k, d]_q$*-code* $\longleftrightarrow$ n*-arc in* $PG(n - k - 1, q)$

Columns of a parity-check matrix $\longleftrightarrow$ *points in* $PG(n - k - 1, q)$

NMDS codes

Definition (Singleton defect)

$$\Delta(\mathcal{C}) = n - k + 1 - d$$

$$\Delta(\mathcal{C}) = 0 \implies \mathcal{C} \quad \text{MDS}$$

$$\Delta(\mathcal{C}) = 1 \implies \mathcal{C} \quad \text{A(lmost)MDS}$$

$$\begin{matrix} \Delta(\mathcal{C}) = 1 \\ \Delta(\mathcal{C}^{\perp}) = 1 \end{matrix} \implies \mathcal{C} \quad \text{N(ear)MDS}$$

NMDS $[n, 3, d]_q$*-code* $\longleftrightarrow$ $(n, 3)$*-arc in* $PG(2, q)$

Columns of a parity-check matrix $\longleftrightarrow$ *points in* $PG(2, q)$

Algebraic constructions

Idea of Segre and Lombardo-Radice

The points of the arc are chosen, with few exceptions, among the points of a conic or a cubic curve

1. Choose a $\mathcal{K} \subset \mathrm{PG}(2,q)$ having a low degree parametrization
2. Prove that $\mathcal{K}$ is an arc
3. $\forall\ P \in \mathrm{PG}(2,q) \setminus \mathcal{K}$ construct $\mathcal{H}_P$ algebraic curve which expresses the collinearity condition between P and $P_1, P_2 \in \mathcal{K}$
4. Show that $\mathcal{H}_P$ is absolutely irreducible for almost all P
5. Use the Hasse-Weil theorem to show that, if q is large enough, then $(\overline{x}, \overline{y}) \in \mathcal{H}_P(\mathbb{F}_q)$: $P_1(\overline{x})$ and $P_2(\overline{y})$ collinear with P
6. Extend $\mathcal{K}$ with some extra points

Example: Construction of arcs in projective planes

$$\mathcal{K} = \{(f(t), g(t)) \mid t \in \mathbb{F}_q\} \subset AG(2, q)$$

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- $P = (a, b)$ covered by $\mathcal{K}$ if there exist $x, y \in \mathbb{F}_q$ with

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  - the algebraic curve $\mathcal{H}_P$ has an $\mathbb{F}_q$-rational point $(\overline{x}, \overline{y})$
  - $(f(\overline{x}), g(\overline{x})) \neq (f(\overline{y}), g(\overline{y}))$, not a pole of x or y

Example: Construction of arcs in projective planes II

$$\mathcal{K} = \{(\underbrace{\overbrace{L(t)+c}^{f(t)}, \overbrace{(L(t)+c)^3}^{g(t)}}_{P_t}) \mid t \in \mathbb{F}_q\}, \qquad -3c \notin \mathrm{Im}(L)$$

$$\mathcal{H}_P: \quad b + (L(x)+c)(L(y)+c)^2 + (L(x)+c)^2(L(y)+c) - a((L(x)+c)^2 + (L(x)+c)(L(y)+t) + (L(y)+c)^2) = 0$$

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(Szőnyi, 1985)

if $b \neq a^3$

- $\mathcal{H}_P$ is absolutely irreducible
- $\mathcal{H}_P$ has at least $q + 1 - 9\deg(L)^2\sqrt{q}$ points

Algebraic constructions

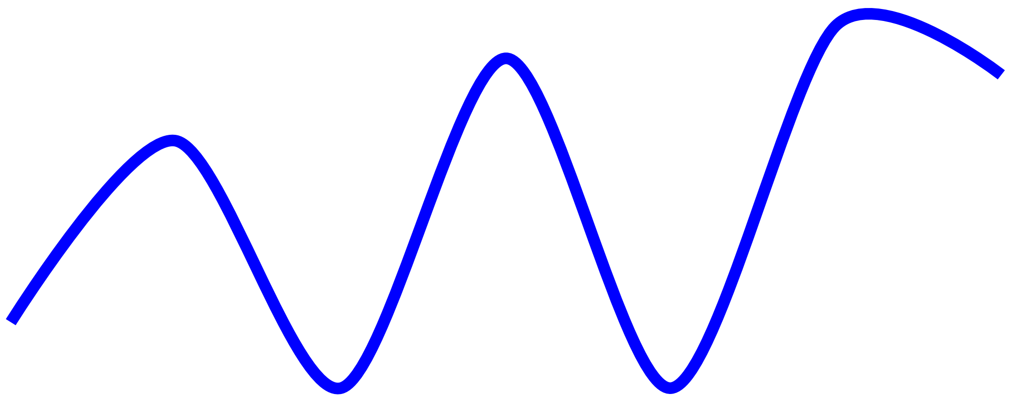
Idea of Segre and Lombardo-Radice

The points of the arc are chosen, with few exceptions, among the points of a conic or a cubic curve

1. $q/2$: Segre, Hirschfeld
2. $q/3$: Abatangelo, Korchmàros, Szőnyi, Voloch
3. $q/4$: Korchmàros
4. $2q^{9/10}$: Szőnyi
5. $cq^{3/4}$: Szőnyi-Voloch-Anbar-B.-Giulietti-Platoni

Infinite families of complete (n, r)-arcs, $r > 2$

- $\mathbb{F}_q$-rational points of irreducible curve of degree r



- 2-character sets in $\mathrm{PG}(2, q)$

$r = 3$

No other examples than irreducible cubics!

Complete $(n, 3)$-arcs from cubic curves

Proposition (Hirschfeld-Voloch)

- $\mathcal{E}$: *plane elliptic curve*
- $j(\mathcal{E}) \neq 0$
- $q \geq 121$

$$\boxed{\mathcal{E} \text{ is a complete } (n,3)\text{-arc in } \mathrm{PG}(2,q)}$$

Proposition (Giulietti)

- $\mathcal{E}$: *plane elliptic curve*
- $|\mathcal{E}|$ *even*
- $j(\mathcal{E}) = 0$
- $q = p^r$, $p > 3$, $q > 9887$
- r *even or* $p \equiv 1 \bmod 3$

$$\boxed{\mathcal{E} \text{ is a complete } (n,3)\text{-arc in } \mathrm{PG}(2,q)}$$

$\mathcal{E}$ *complete* $(n,3)$*-arc* $\Longrightarrow q - 2\sqrt{q} + 1 \leq |\mathcal{E}| \leq q + 2\sqrt{q} + 1$

Complete $(n, 3)$-arcs

UPPER and LOWER BOUNDS

$\mathcal{A}$: *complete* $(n, 3)$*-arc*

$$\sqrt{6(q+1)} \leq |\mathcal{A}| \leq 2q + 1$$

Random construction

$$q \leq 30000$$

$$|\mathcal{A}| \simeq \sqrt{6q} \log q$$

Algebraic constructions of small complete $(n, 3)$-arcs

Idea of Segre and Lombardo-Radice

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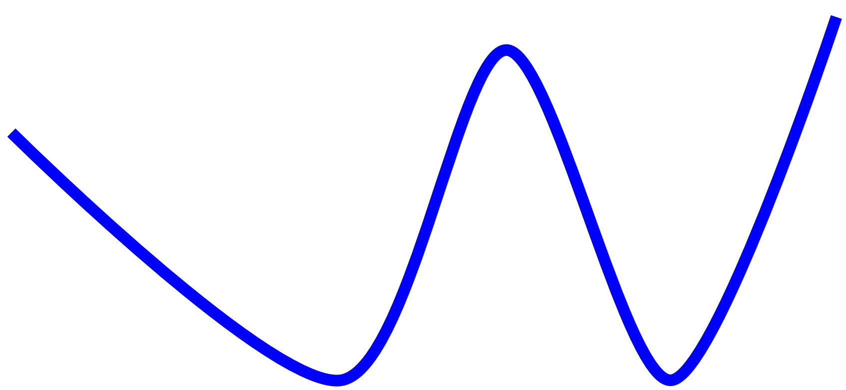
Our Idea

The points of the $(n, 3)$-arc are chosen, with few exceptions, among the points of an irreducible quartic curve

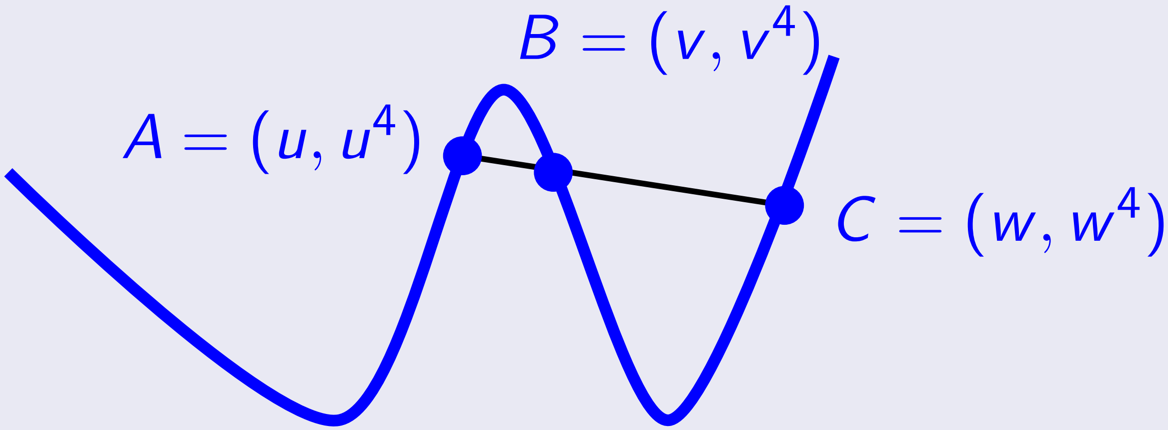
Small complete $(n, 3)$-arcs from quartic curves

- p: odd prime, $p \equiv 2 \bmod 3$
- $\sigma = p^{h'}$, h' odd
- $q = p^h$, $h > h'$, $h' \mid h$



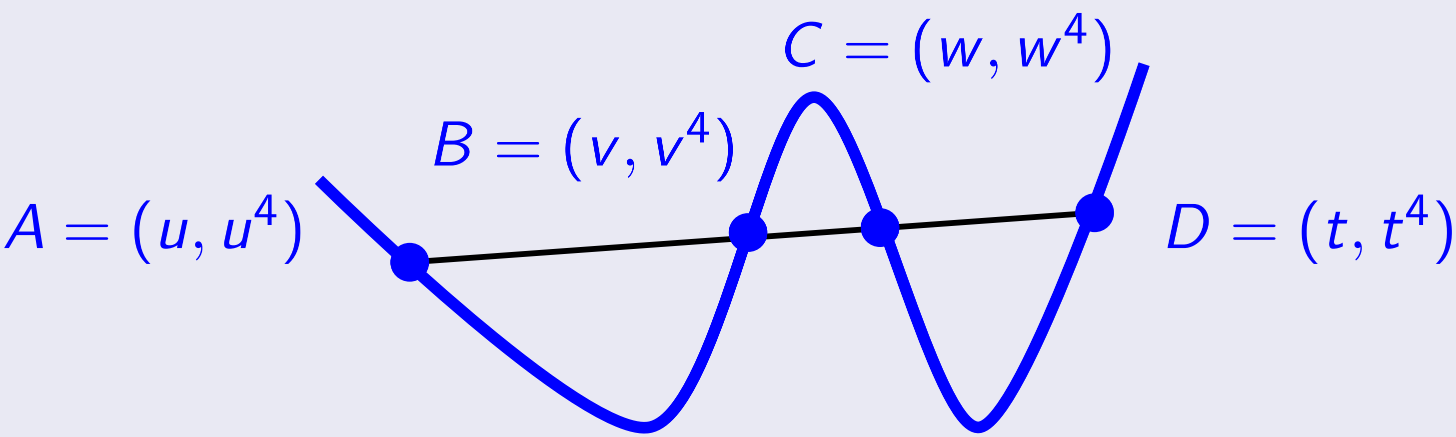
$\mathcal{Q} = \{(x, x^4) \mid x \in \mathbb{F}_q\}$

Proposition



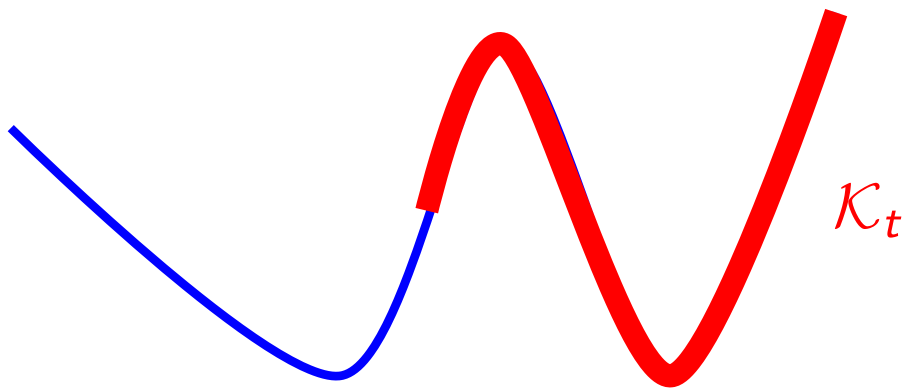
COLLINEAR $\iff$ $u^2 + v^2 + w^2 + uv + uw + vw = 0$

Proposition



COLLINEAR $\iff$ $\begin{cases} u^2 + v^2 + w^2 + uv + uw + vw = 0 \\ u + v + w + t = 0 \end{cases}$

- $M := \{(a^\sigma - a) \mid a \in \mathbb{F}_q\}$ $\qquad M \simeq \mathbb{F}_{\frac{q}{\sigma}} \leq (\mathbb{F}_q, +)$
- $\mathcal{K}_t := \{(v, v^4) \mid v \in M + t\}$, with $t \notin M$



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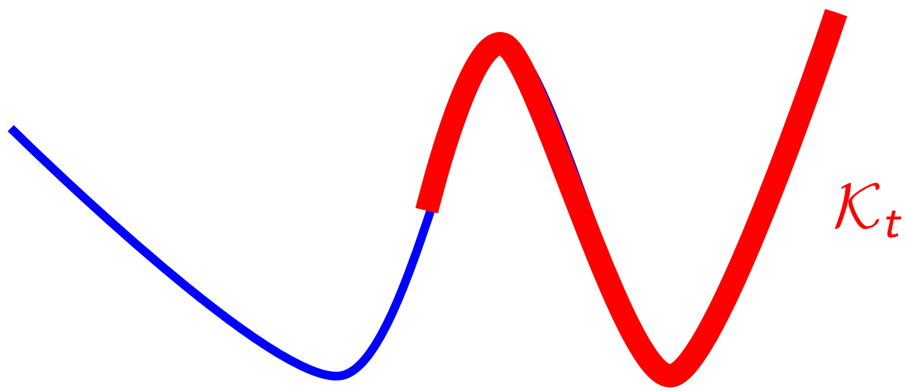


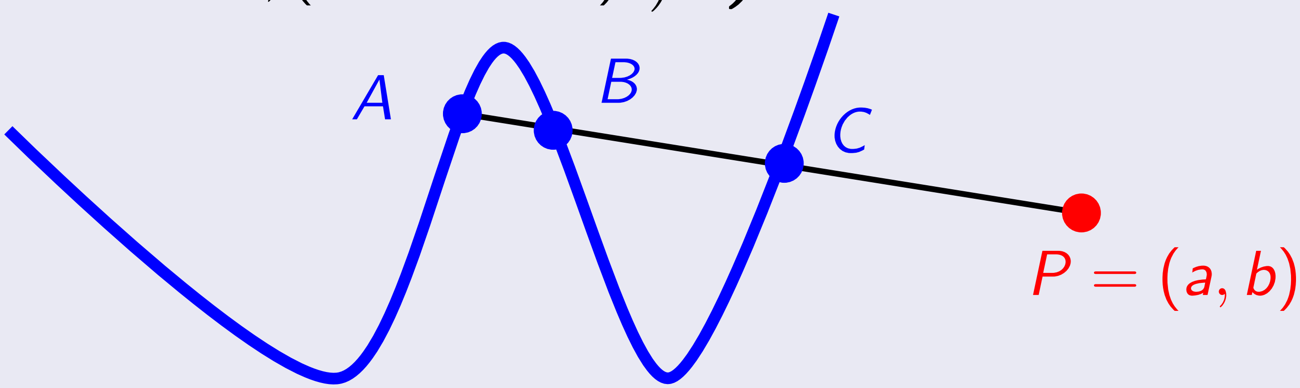
Proposition

$\mathcal{K}_t$ is a $(k, 3)$-arc.

Points off $\mathcal{Q}$

Proposition

$$\left.\begin{array}{l} A = \left(x^\sigma - x + t, (x^\sigma - x + t)^4\right) \\ B = \left(y^\sigma - y + t, (y^\sigma - y + t)^4\right) \\ C = \left(z^\sigma - z + t, (z^\sigma - z + t)^4\right) \end{array}\right\} \in \mathcal{K}_t \quad \text{and} \quad \begin{array}{c} P = (a, b) \in \\ \mathrm{AG}(2, q) \setminus \mathcal{Q} \end{array}$$



COLLINEAR

$$\Updownarrow$$

$$\left\{\begin{array}{l} (z^\sigma - z)^2 + (z^\sigma - z)((x^\sigma - x) + (y^\sigma - y) + 4t) + 4t(x^\sigma - x + y^\sigma - y) + \\ +6t^2 + (x^\sigma - x)(y^\sigma - y) + (x^\sigma - x)^2 + (y^\sigma - y)^2 = 0 \\ \\ a((x^\sigma - x)^2 + (y^\sigma - y)^2 + 2t^2 + 2t(x^\sigma - x) + \\ +2t(y^\sigma - y))(x^\sigma - x + y^\sigma - y + 2t) - (x^\sigma - x + t)(y^\sigma - y + t)\cdot \\ \cdot((x^\sigma - x)^2 + (y^\sigma - y)^2 + (x^\sigma - x)(y^\sigma - y) + 3t^2 \\ +3t(x^\sigma - x + y^\sigma - y)) - b = 0 \end{array}\right.$$

$\mathcal{H}_P$

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for almost all $P \in \mathrm{AG}(2, q) \setminus \mathcal{Q}$
the space curve $\mathcal{H}_P$ is absolutely irreducible
and it has genus $g \leq 30\sigma^3 - 12\sigma^2 - 4\sigma + 1$

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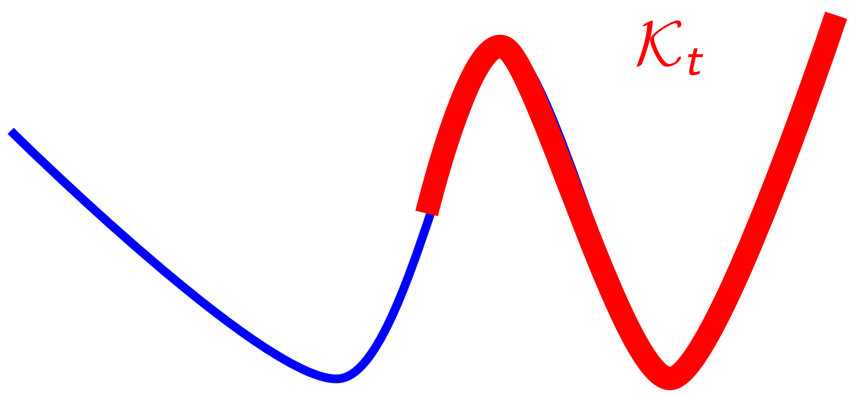
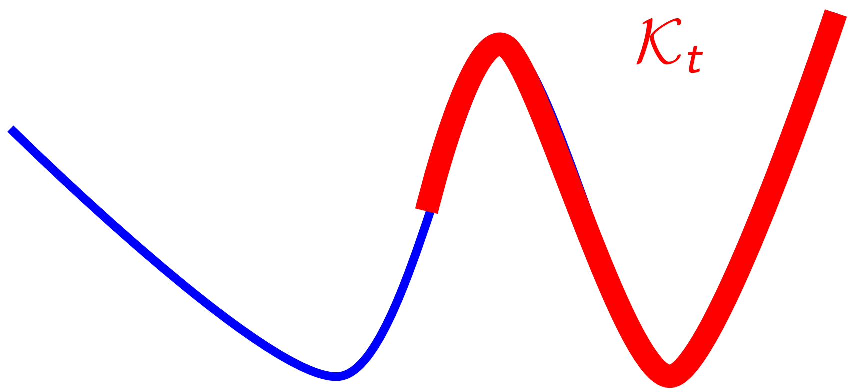


Theorem

$q \geq 3600\,\sigma^6$

$\mathcal{K}_t$ is a 3-arc covering $\mathrm{AG}(2,q) \setminus \mathcal{Q}$ (except possibly $Y = 0$)

Points of $\mathcal{Q}$

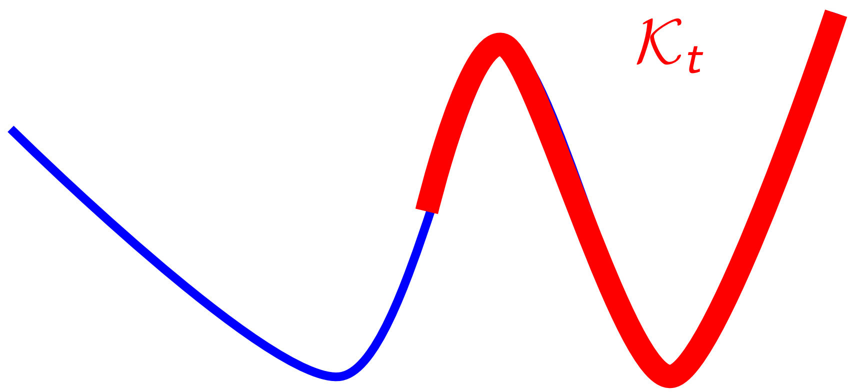


Problem

To find $T \subset \mathcal{Q}$

- T *is a 3-arc*
- T *contains at least one coset* $\mathcal{K}_t$
- T *covers all the points of* $\mathcal{Q} \setminus T$

Points of $\mathcal{Q}$



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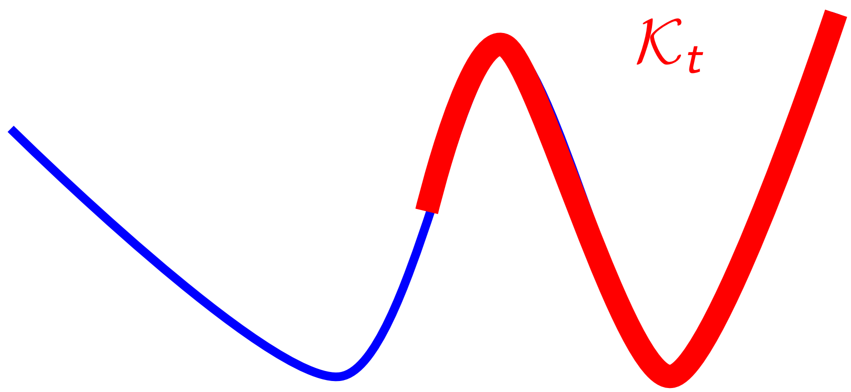
In particular

- 4 points of T are not collinear
- every point in $\mathcal{Q} \setminus T$ is collinear with 3 points of T

Points of $\mathcal{Q}$



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Solution

Use 4-independent subsets!

k-independent subsets

Definition

$\mathcal{G}$: abelian group, $A \subset \mathcal{G}$

- $A \subset \mathcal{G}$ *k-independent subset* $\iff$ $x_1 + x_2 + \cdots + x_k \neq 0$ $\forall\, x_i \in A$
- $g \in \mathcal{G} \setminus A$ *covered by* A $\iff$ $x_1 + x_2 + \cdots + x_{k-1} + g = 0$ *for some* $x_i \in A$
- A *maximal k-independent subset* $\iff$ $\forall\, g \in \mathcal{G} \setminus A$ g *is covered by* A

3-independent subsets

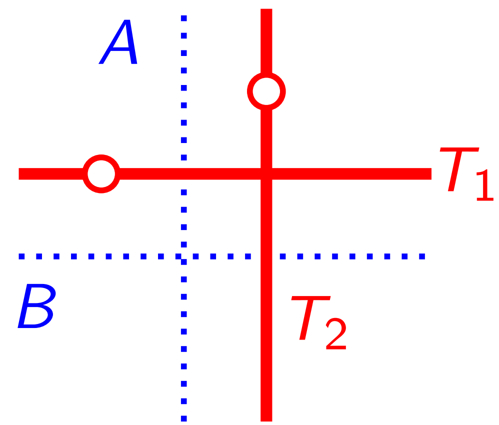
Proposition

G abelian, not elementary 3-abelian
$T \subset G$ maximal 3-independent subset

$$c_1\sqrt{|G|} \le |T| \le \frac{|G|}{2}$$

- (Voloch) $p \equiv 2 \bmod 3 \implies T = \left\{\pm 1, \pm 3, \ldots, \pm\frac{p-2}{3}\right\} \subset \mathbb{Z}_p$
- (Voloch) $p \equiv 1 \bmod 3 \implies T = \left\{-1, 1, 3, 4, \ldots, \frac{p-1}{3}\right\} \subset \mathbb{Z}_p$
- (Szőnyi) $G = A \times B$, A, B not elementary 3-abelian

$$T_1 = \{(a,x) | x \in B, x \neq -2b\}$$
$$T_2 = \{(y,b) | y \in A, y \neq -2a\}$$



4-independent subsets

- $M := \{(a^\sigma - a) \mid a \in \mathbb{F}_q\}$
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4-independent subset in $\mathbb{F}_q/M \equiv \mathbb{F}_\sigma$

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Proposition

$\mathcal{T}$: *4-independent subset of* $\mathbb{F}_q/M$

$$\mathcal{K}_\mathcal{T} = \bigcup_{M+t\in\mathcal{T}} \mathcal{K}_t$$

$\mathcal{K}_\mathcal{T}$ *is a* $(k, 3)$*-arc*

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$\sigma = p^{h'}$, h' odd $\Longrightarrow$ $\sigma = p,\ \sigma = p^3,\ \ldots$

Theorem

- $\mathcal{T}$: *4-independent subset of* $\mathbb{F}_q/M$
- $|\mathcal{T}| = n$
- $|\mathbb{F}_q/M \setminus Cov(\mathcal{T})| \leq m$
- $\mathcal{K}_{\mathcal{T}} = \bigcup_{M+t\in\mathcal{T}} \mathcal{K}_t$
- $q \geq 3600\,\sigma^6$

$\exists\ \mathcal{K}$ *complete 3-arc*

$$\mathcal{K}_{\mathcal{T}} \subset \mathcal{K} \subset \mathcal{Q}$$

$$|\mathcal{K}| \leq (n+m)\frac{q}{\sigma} + 6$$

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- $\begin{array}{c} \sigma = p \\ p \equiv 1 \bmod 4 \\ p \geq 29 \end{array} \Longrightarrow \begin{array}{c} n = \frac{p-5}{4} \\ m = 1 \end{array} \Longrightarrow |\mathcal{K}| \lesssim \frac{q}{4}$
- $\begin{array}{c} \sigma = p \\ p \equiv 3 \bmod 4 \\ p > 29 \end{array} \Longrightarrow \begin{array}{c} n = \frac{p-7}{4} \\ m = 3 \end{array} \Longrightarrow |\mathcal{K}| \lesssim \frac{q}{4}$
- $\sigma \geq p^3 \Longrightarrow \begin{array}{c} n = 2\sqrt{\frac{\sigma}{p}} + p - 4 \\ m = 2\left(\sqrt{\sigma p} - \sqrt{\frac{\sigma}{p}}\right) \end{array} \Longrightarrow |\mathcal{K}| \lesssim 2\sqrt{\frac{p}{\sigma}}q$

$$|\mathcal{K}| \leq (n+m)\frac{q}{\sigma} + 6$$

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$$\begin{array}{c} \sigma = p^3 \\ p > 13 \\ q = \sigma^7 \end{array} \quad \Longrightarrow \quad |\mathcal{K}| \simeq q^{20/21}$$

Future developments

Idea of Segre and Lombardo-Radice

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Our Idea

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(n, r)-arcs?

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NATURAL IDEA:

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PROBLEM:

The curve $\mathcal{H}_P$ is more complicated

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The curve $\mathcal{H}_P$ is more complicated

$r = 4$: Collinearity condition between

$$P = (a, b),\ P_1 = (u, u^5)\ P_2 = (v, v^5),\ P_3 = (w, w^5),\ P_4 = (s, s^5)$$

$$\begin{cases} b + uv(u^3 + u^2 v + uv^2 + v^3) - a(u^4 + u^3 v + u^2 v^2 + uv^3 + v^4) = 0 \\ w^3 + w^2(u+v) + w(u^2 + uv + v^2) + (u^3 + u^2 v + uv^2 + v^3) = 0 \\ s^2 + s(u+v+w) + u^2 + v^2 + w^2 + uv + uw + vw = 0 \end{cases}$$

THANK YOU
FOR YOUR ATTENTION!